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Topic: FD for time dep problems

Objectives

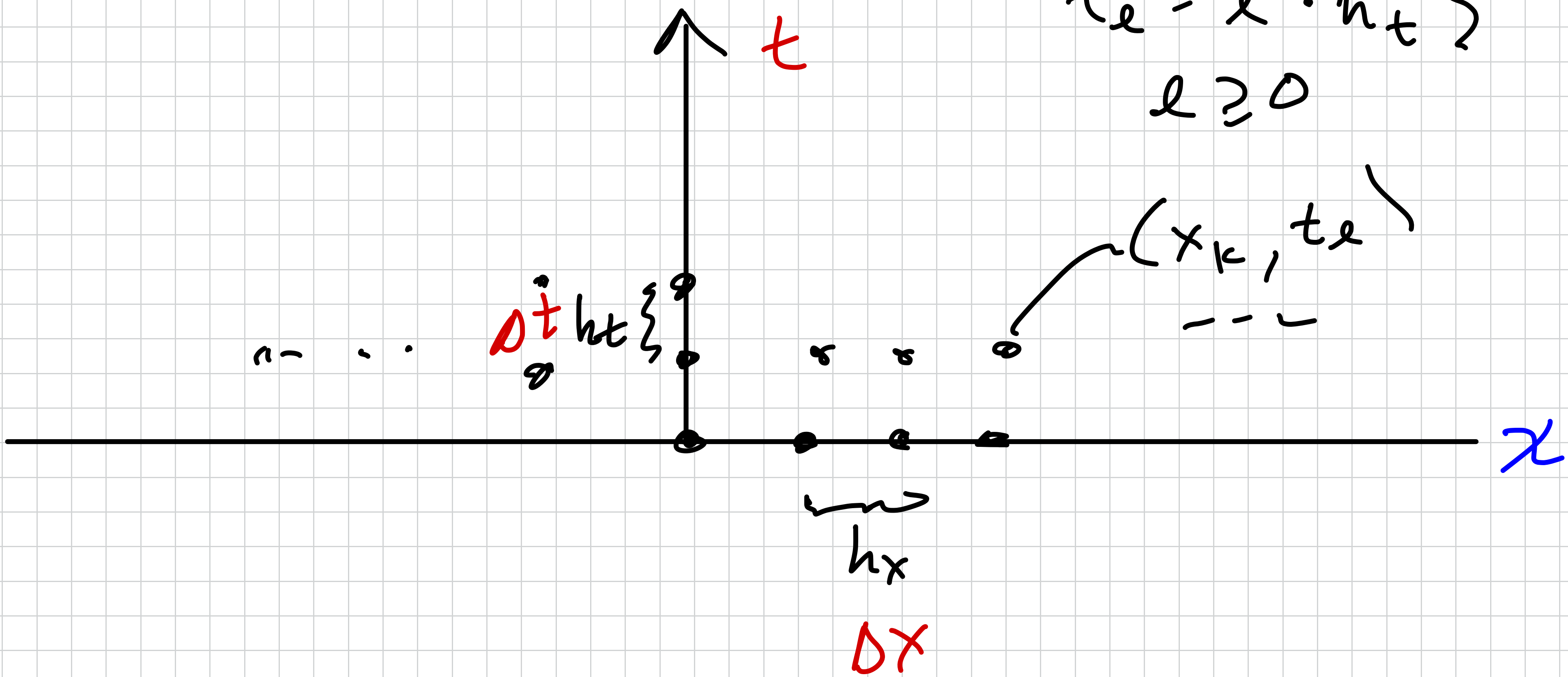
(1) Introduce explicit methods
implicit methods

(2) Develop a 2-level scheme

(3) Say something about error

Grid: $\{(x_k, t_\ell) : x_k = k \cdot h_x,\ t_\ell = \ell \cdot h_t\}$

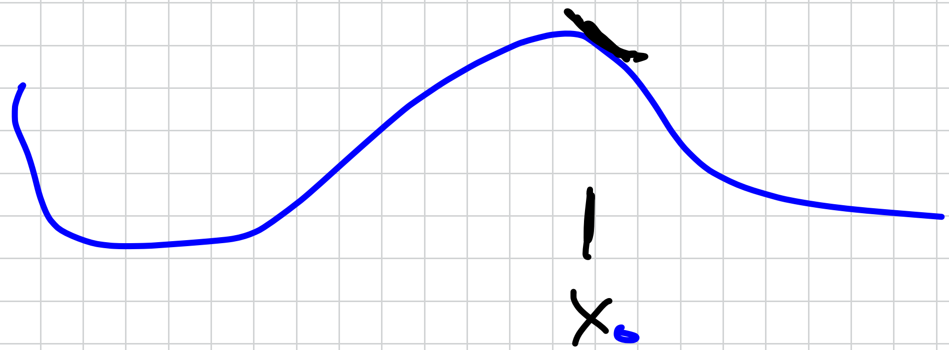
$\ell \geq 0$



advection $\begin{cases} u_t + c\, u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$

→ used to approximate derivatives

Consider $f(x)$:



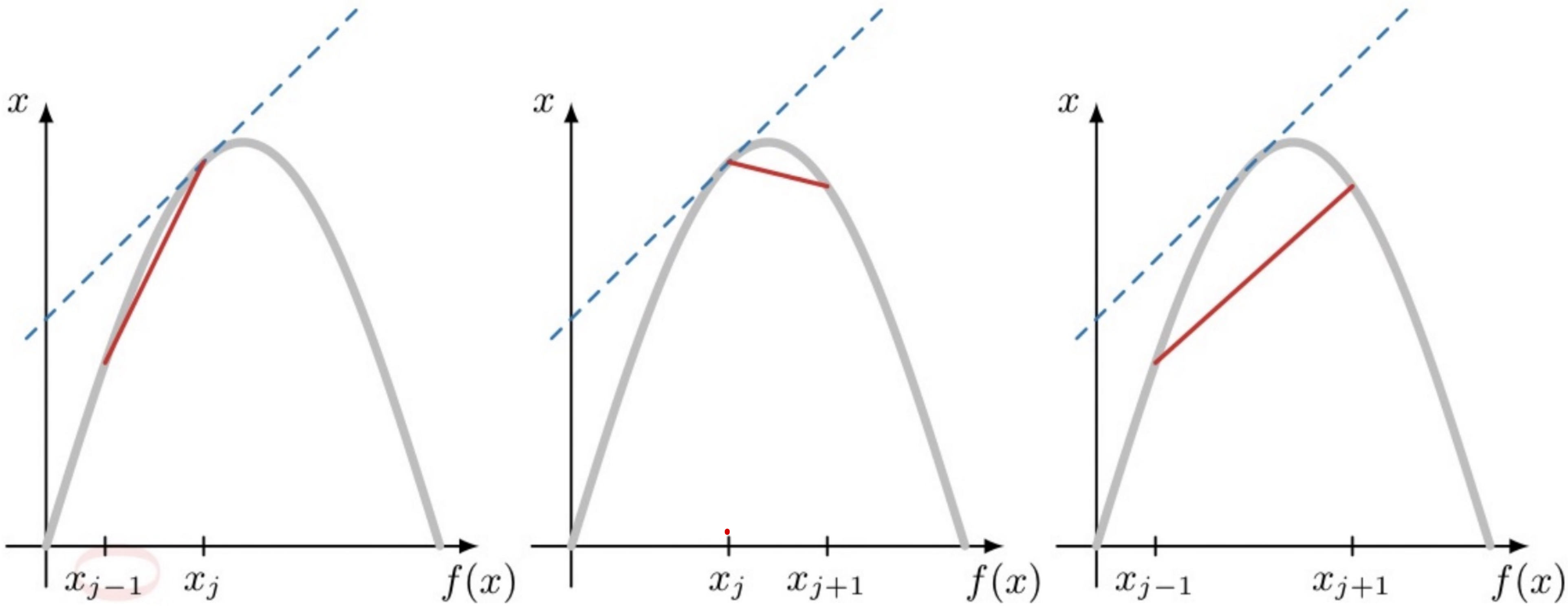
Taylor:

$$f(x_0+h_x) = f(x_0) + f'(x_0)\, h_x + f''(\xi)\frac{h_x^2}{2}$$

$$\Rightarrow \quad f'(x_0) = \frac{f(x_0+h_x) - f(x_0)}{h_x} - f''(\xi)\frac{h_x}{2}$$

"forward diff"

"1st order"



Bkd

Fwd

Central

$$f(x_0 + h_x) = \cancel{f(x_0)} + f'(x_0)h_x + \cancel{f''(x_0)\frac{h_x^2}{2}} + f'''(\xi^+)\frac{h_x^3}{6}$$

$$f(x_0 - h_x) = \cancel{f(x_0)} - f'(x_0)h_x + \cancel{f''(x_0)\frac{h_x^2}{2}} - f'''(\xi^-)\frac{h_x^3}{6}$$

$$\frac{f(x_0+h_x) - f(x_0-h_x)}{2h_x} = f'(x_0) + f'''(\xi^+)\frac{h_x^2}{12} + f'''(\xi^-)\frac{h_x^2}{12}$$

"central"

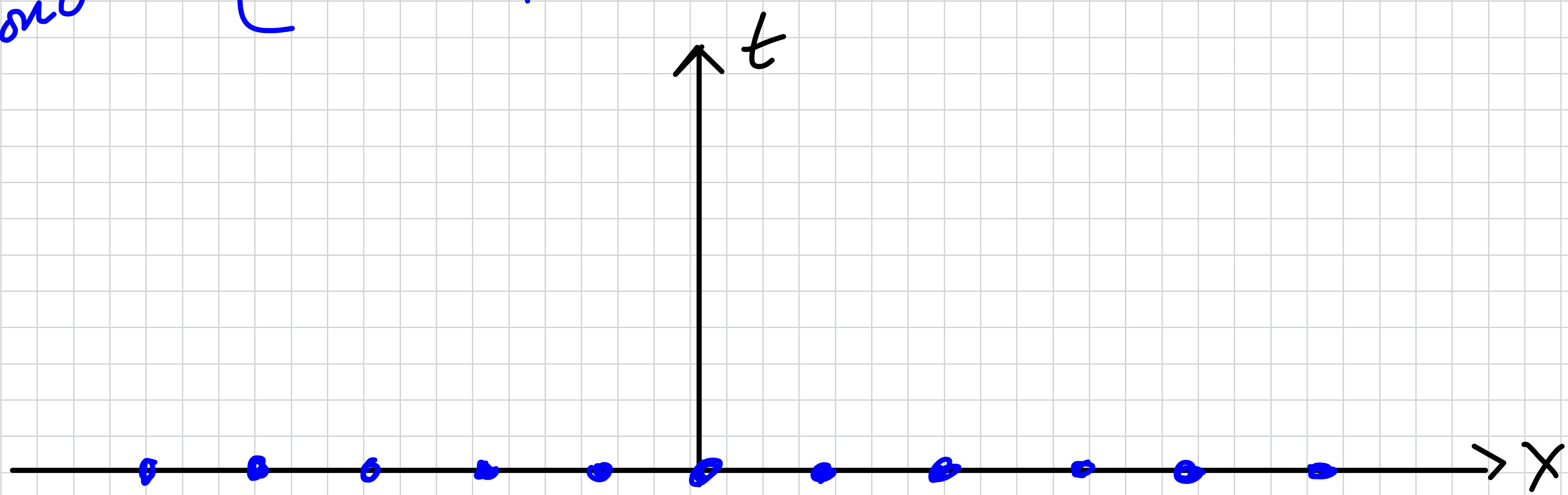
$$O(h_x^2)$$

"2nd order"

$$u_t + a u_x = 0 \qquad a > 0$$

advection with initial condition

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + a \dfrac{\partial u(x,t)}{\partial x} = 0 \\ u(x,0) = u_0(x) \end{cases}$$



initial data $u_0(x)$

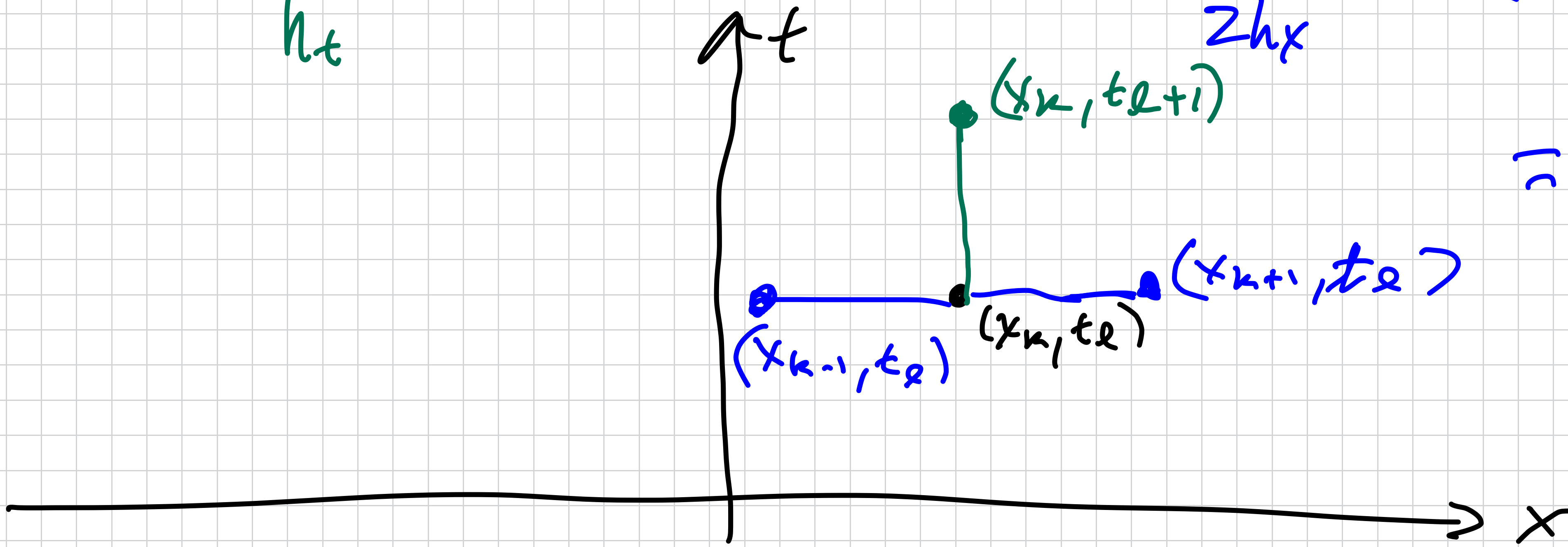
$u(1) = u(0)$

$$\frac{\partial u(x,t)}{\partial t} + a \, \frac{\partial u(x,t)}{\partial x} = 0$$

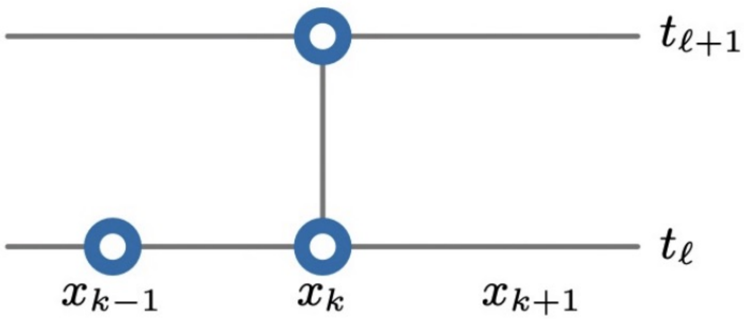
Fwd differencing in time at $x = x_k$

central differencing in space at time t_ℓ

$$\frac{u(x_k, t_\ell + h_t) - u(x_k, t_\ell)}{h_t} + a \cdot \frac{u(x_k + h_x, t_\ell) - u(x_k - h_x, t_\ell)}{2h_x} = 0$$

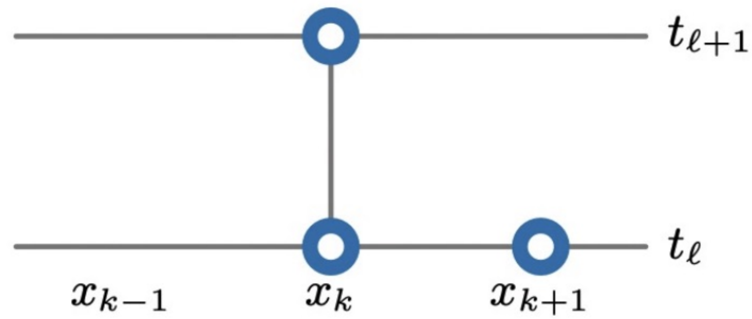


$$\frac{u_{k,\ell+1}-u_{k,\ell}}{h_t}+a\frac{u_{k,\ell}-u_{k-1,\ell}}{h_x}=0$$



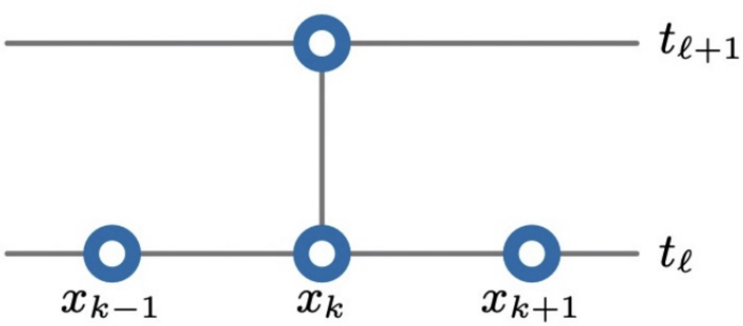
a. Explicit time, backward space (ETBS)

$$\frac{u_{k,\ell+1}-u_{k,\ell}}{h_t}+a\frac{u_{k+1,\ell}-u_{k,\ell}}{h_x}=0$$



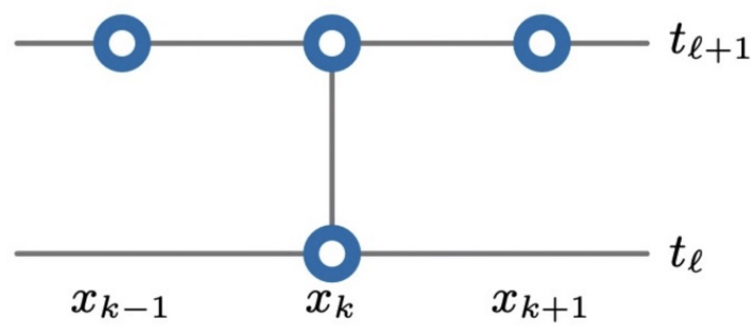
b. Explicit time, forward space (ETFS)

$$\frac{u_{k,\ell+1}-u_{k,\ell}}{h_t}+a\frac{u_{k+1,\ell}-u_{k-1,\ell}}{2h_x}=0$$



c. Explicit time, centered space (ETCS)

$$\frac{u_{k,\ell+1}-u_{k,\ell}}{h_t}+a\frac{u_{k+1,\ell+1}-u_{k-1,\ell+1}}{2h_x}=0$$



d. Implicit time, centered space (ITCS)

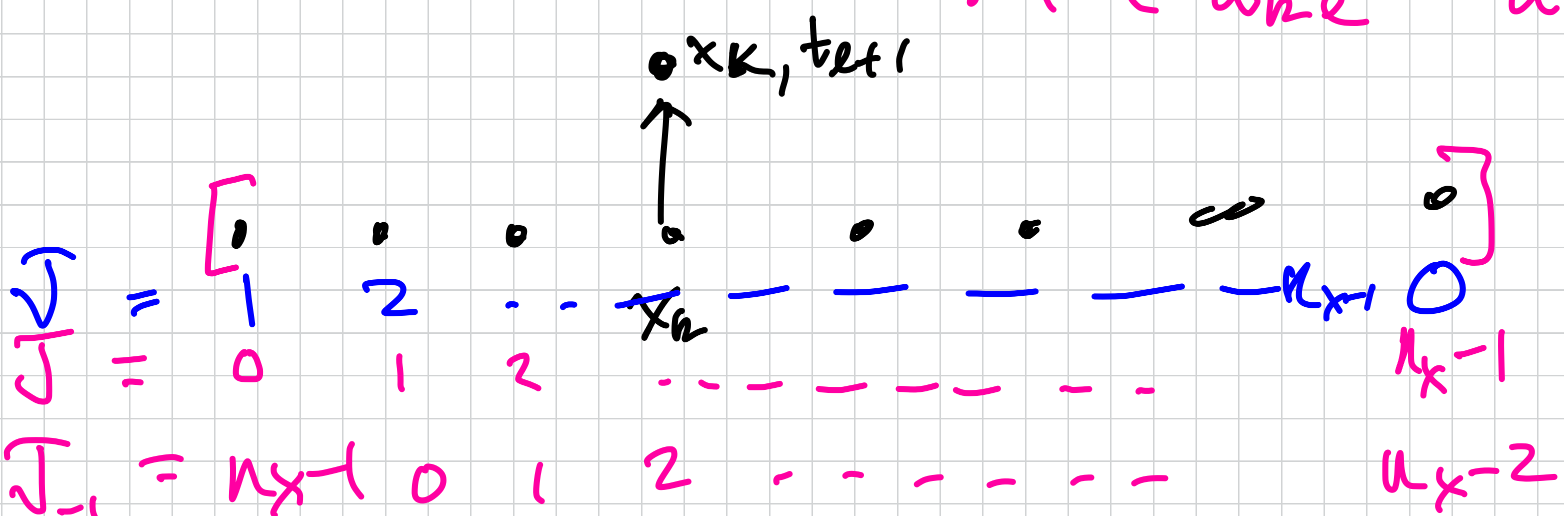
ETBS

$$\frac{u_{k,l+1}-u_{k,l}}{h_t} + a\cdot\frac{u_{k,l}-u_{k-1,l}}{h_x} = 0$$

$$\rightarrow\quad u_{k\,l+1} - u_{k\,l} + \frac{a h_t}{h_x}\left(u_{k\,l} - u_{k-1\,l}\right) = 0$$

$$\rightarrow\quad u_{k\,l+1} = u_{k\,l} - \frac{a h_t}{h_x}\left(u_{k\,l} - u_{k-1\,l}\right)$$

$$= u_{k\,l} - \lambda\left(u_{k\,l} - u_{k-1\,l}\right)$$



Open Questions

1. Why does the profile "smooth"
   
2. " " " " get smaller?
3. Is it traveling at the correct speed?
4. Is the approximation accurate?
   in terms of h_t, h_x
5. Why does the approx. "blow up" with $\lambda > 1$?

TODO: read Remark 5.2

Terminology for FD stencils.

let $\underline{u}_\ell = \begin{bmatrix} \vdots \\ u_{-1,\ell} \\ u_{0,\ell} \\ u_{1,\ell} \\ \vdots \end{bmatrix}$

let $e_{k,\ell} = u(x_k, t_\ell) - u_{k,\ell}$

exact solution to $u_t + a u_x = 0$ at (x_k, t_ℓ)

approximation to $u(x,t)$ at (x_k, t_ℓ)

$\Rightarrow$ look at $\underline{e}_\ell = \underline{U}_\ell - \underline{u}_\ell$

exact vector

approximate vector

Definition 5.7: Two-Level Linear Finite-Difference Scheme

A finite-difference scheme that can be written as,

$$P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell + h_t \boldsymbol{b}_\ell, \tag{5.5}$$

is called a two-level linear finite-difference scheme. Each iteration depends only on two instances of time. Examples are given in Example 5.8.