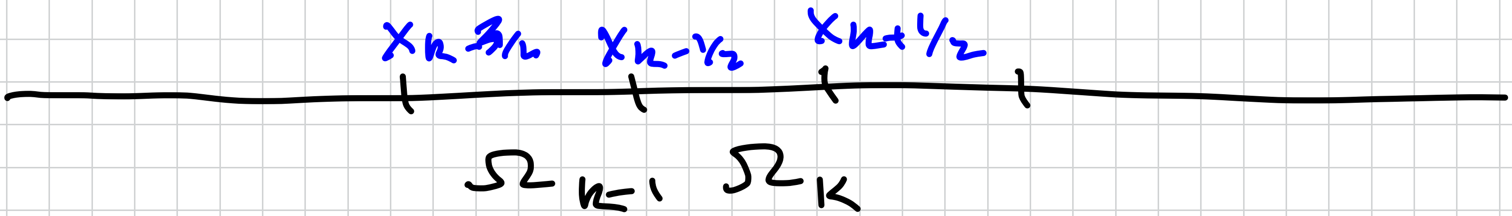
Today

- High(er) order FV methods

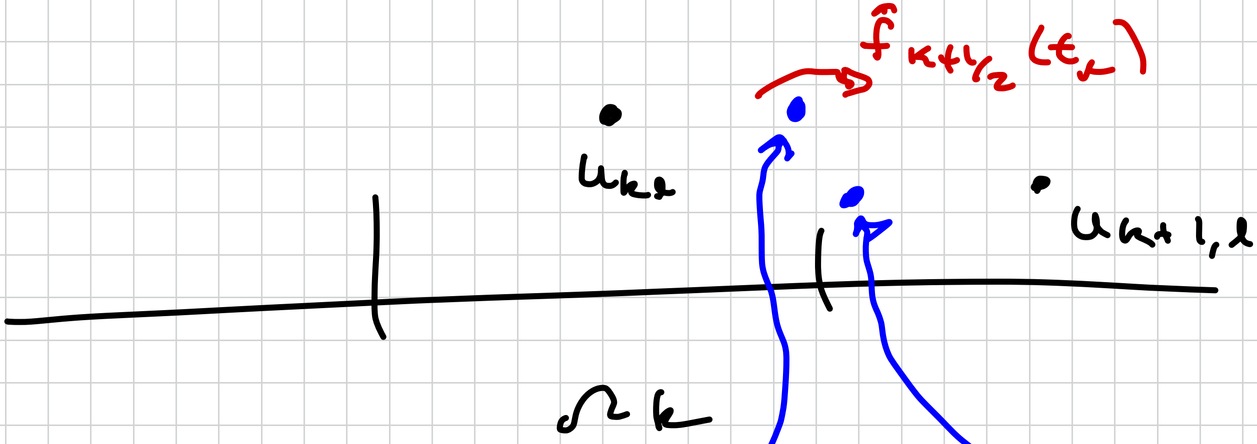
Recap



$$u_{k,\ell} = \overline{u_k}(t_\ell)$$

↑ the average of u in cell Ω_k

let $\hat{f}_{k+1/2}(t)$ be the numerical flux



look for $f^*(u^-_{k+1/2}, u^+_{k+1/2})$

for $u_t + a\,u_x = 0$

let $u^-_{k+1/2,\ell} = u_{k\ell}$

$u^+_{k+1/2,\ell} = u_{k+1\,\ell}$

$\boxed{\text{for } a>0}$

for any $a \in \mathbb{R}$

$f(u) = au$ advection

$$f^*_{k+1/2} = \frac{a\,u_{k\ell} + a\,u_{k+1\ell}}{2} - \frac{|a|}{2}\left(u_{k+1\ell} - u_{k\ell}\right)$$

$= F.O.U.$ (first-order upwind)

what about

$$u_t + \left(f(u)\right)_x = 0 \; ?$$

$$\Rightarrow \quad u_t + \underbrace{f'(u)}_{\text{looks like "a"}} u_x = 0$$

$$u_t + a\, u_x = 0 \qquad \text{ETBS:}$$

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + \overset{f'(u)}{a}\, \frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0$$

Burgers: $u_t + u\, u_x = 0$

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + u_{k,\ell}\, \frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0$$

FV:

$$\frac{u_{k\ell+1} - u_{k\ell}}{h_t} + \frac{f^*_{k+1/2} - f^*_{k-1/2}}{h_x} = 0$$

$$f^*_{k+1/2} = f^*(u_{k\ell}, u_{k+1\ell})$$

$$= \frac{f(u_{k\ell}) + f(u_{k+1\ell})}{2} - \frac{\alpha_{k+1/2}}{2}(u_{k+1\ell} - u_{k\ell})$$

$$\alpha_{k+1/2} = \max\left(|f'(u_{k\ell})|, |f'(u_{k+1\ell})|\right)$$

"local" Lax-Friedrichs method or flux

LLF

So far: ✓ method for nonlinear $f(u)$

✓ linear in accuracy

around §6.3 what higher order accuracy?

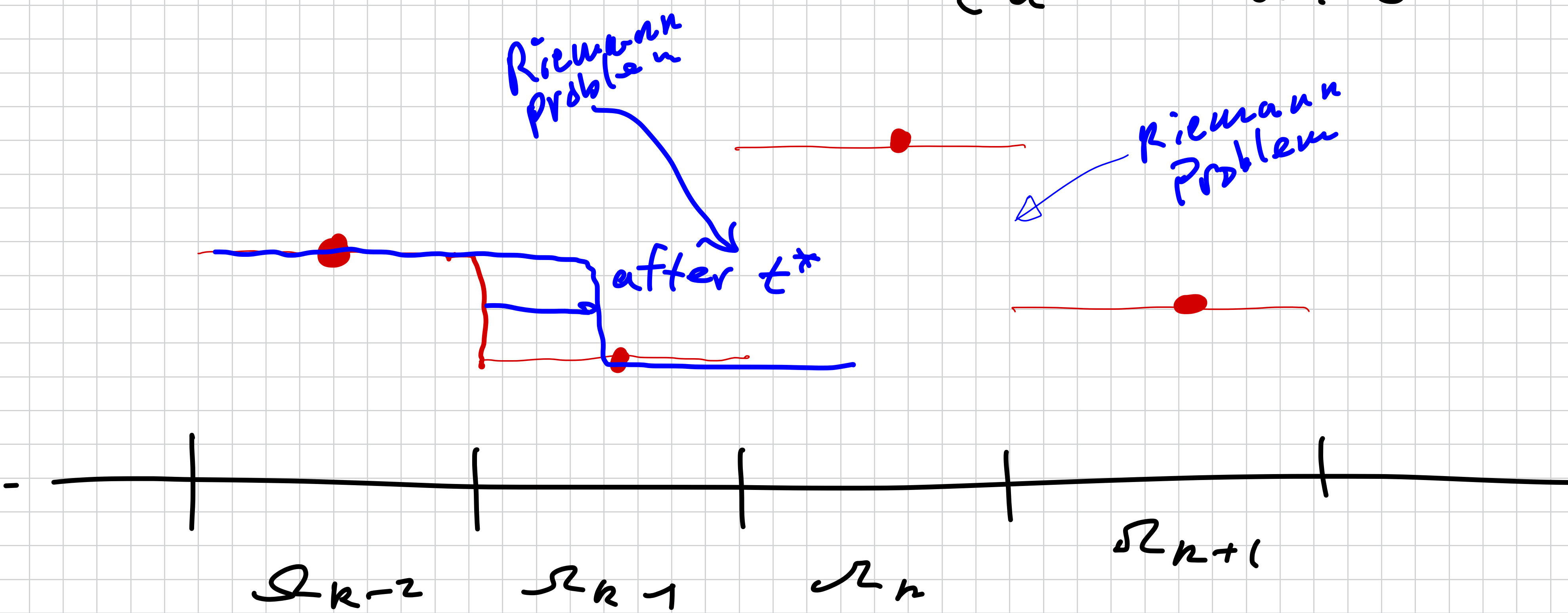
what about systems of PDE?

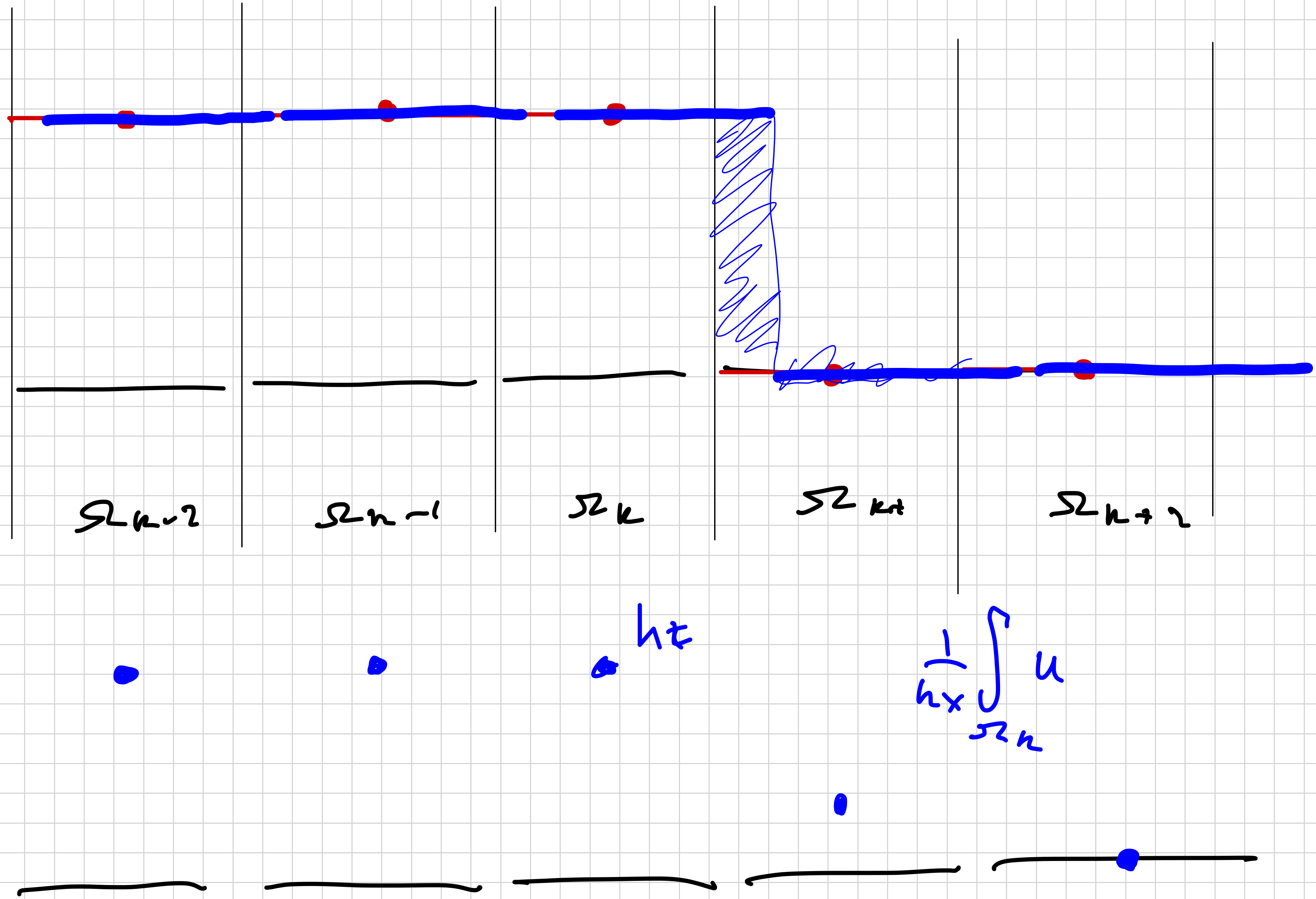
what about 2D / 3D?

Godunov's Method

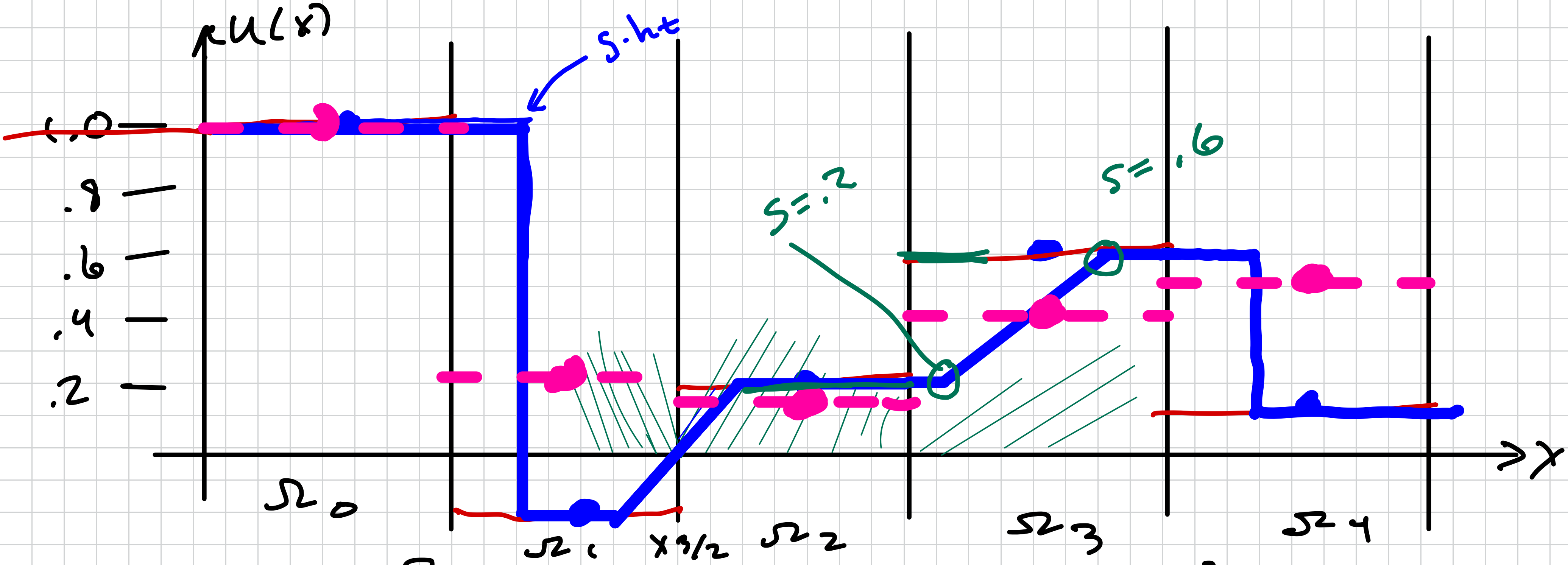
Consider $u_t + \left(\frac{u^2}{2}\right)_x = 0$

Riemann Problem $$u(x,0) = \begin{cases} u^- & u \le 0 \\ u^+ & u > 0 \end{cases}$$





Ω_{k-2}
Ω_{k-1}
Ω_k
Ω_{k+1}
Ω_{k+2}
h_t
$\frac{1}{h_x}\int_{\Omega_k} u$



$$u(x,0) = \begin{cases} 1 & x \in \Omega_0 \\ -.2 & x \in \Omega_1 \\ .2 & x \in \Omega_2 \\ .6 & x \in \Omega_3 \\ .2 & x \in \Omega_4 \end{cases}$$

$$f(u) = \frac{u^2}{2}$$

at $x_{1/2}$: $s = \dfrac{\frac{(-.2)^2}{2} - \frac{(1.0)^2}{2}}{-.2 - 1.0}$

$$= \frac{\frac{-.96}{2}}{-1.2} = \frac{-.48}{-1.2} = .4$$

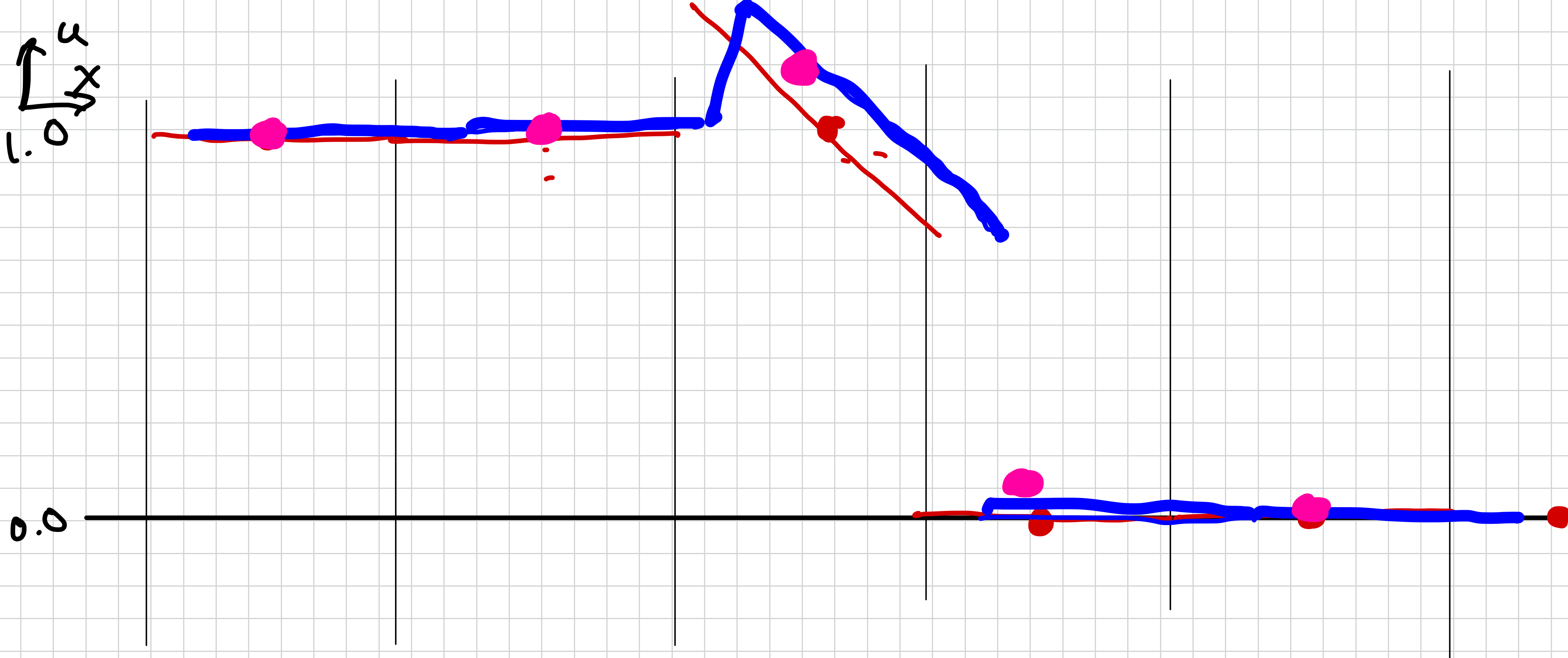
Theorem 6.4: Rankine-Hugoniot relation

Let $\widehat{x}(t)$ be a curve describing a jump discontinuity in a weak solution of the 1D conservation law in Equation (6.2). Then,

$$\widehat{x}'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-}, \tag{6.18}$$

where u^- and u^+ are the values of $u(x,t)$ to the left and to the right of the jump discontinuity.

- **R**econstruct solution
- **E**volve the solution (as Riemann)
- **A**verage solution



$$u(x) = m\,(x - x_{mid}) + u_{average}$$