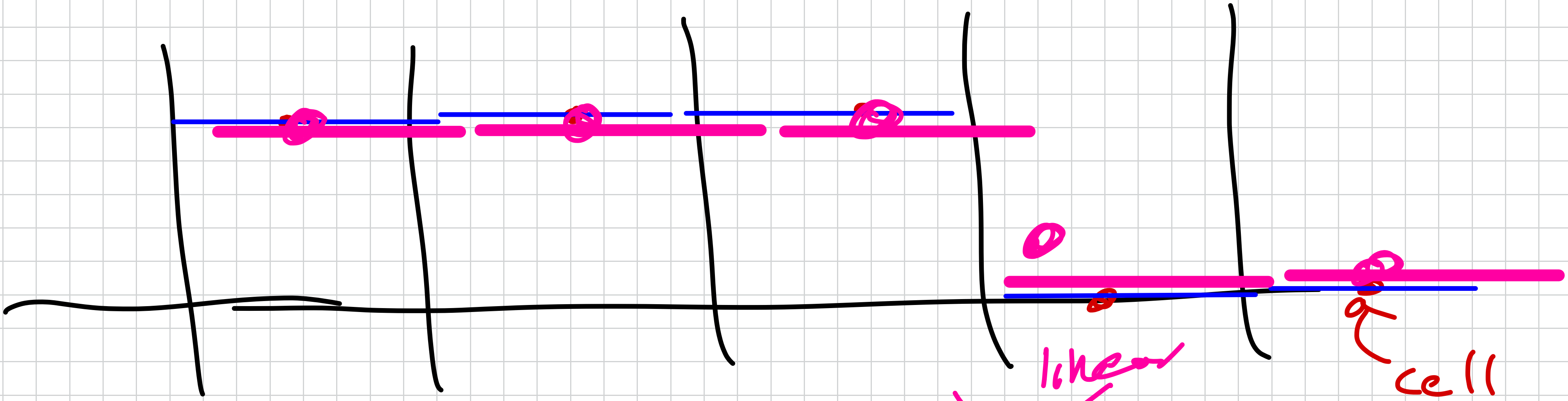
Today 2/14



① Reconstruct a ~~constant~~ linear polynomial average

② evolve the Riemann Problem

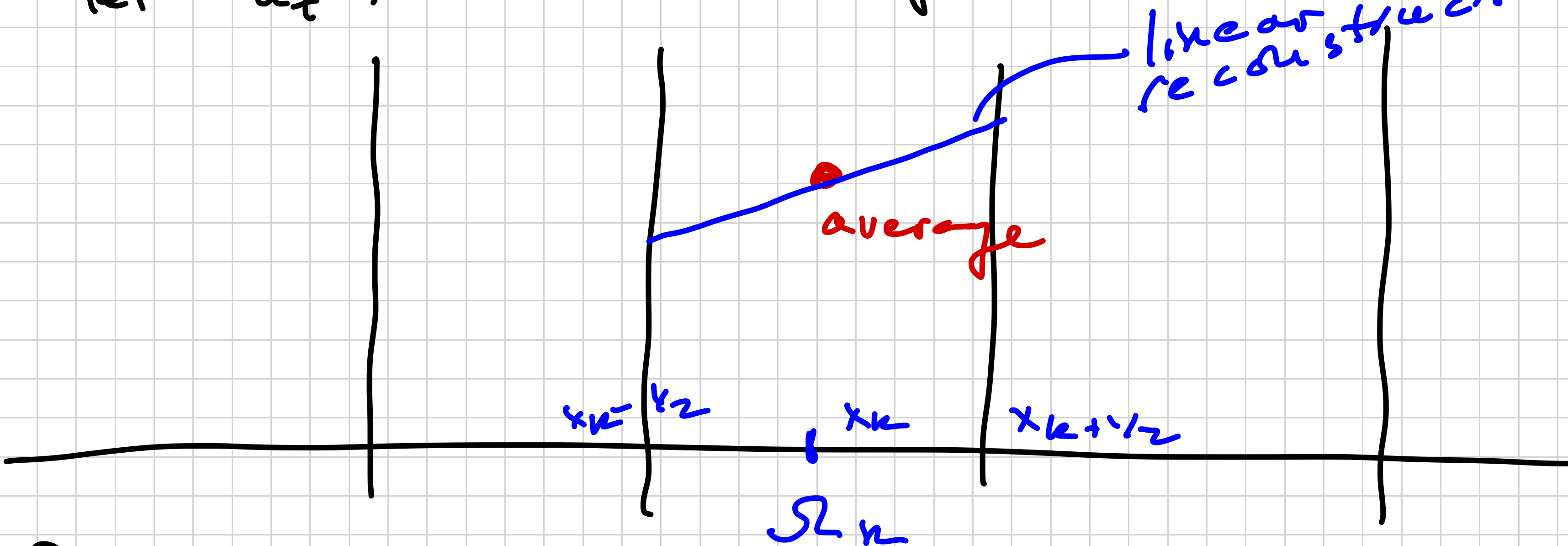
③ compute averaged

GOTO ①

Where are we at?

- linear, nonlinear scalar problems
  - ↳ use F.O.U. or Godunov or LLF
  - "first order"
- for higher order:
  - linear PDE → use REA on linear reconstruction
  - nonlinear PDE → Godunov, but approximate the Riemann problem
- systems later

let $u_t + a u_x = 0$ + periodic



① u_k = average

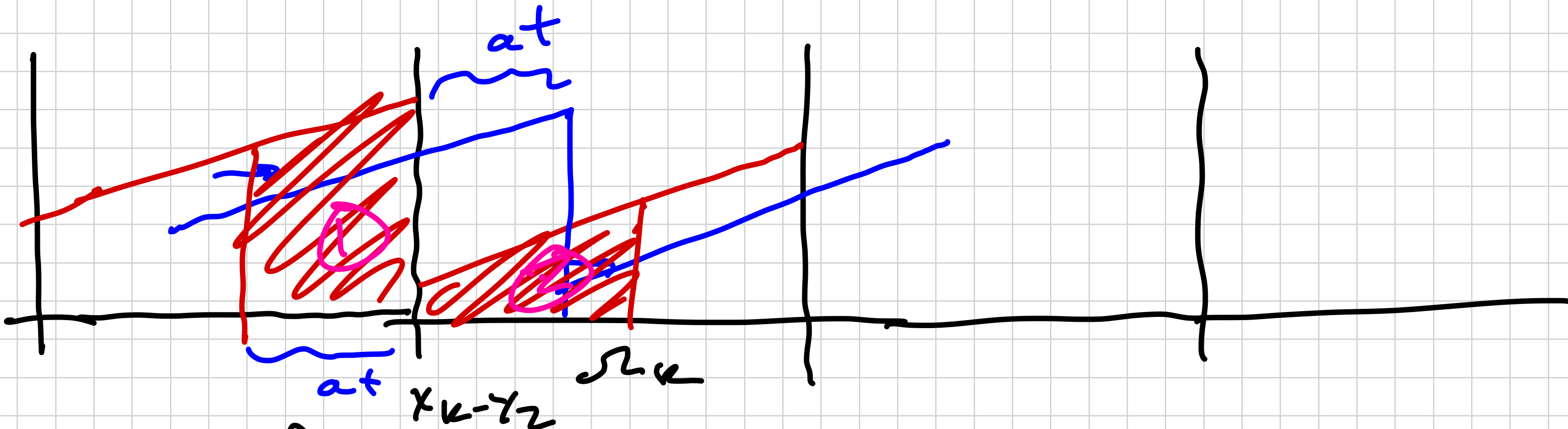
$q_k(x)$ = linear

$$= u_k + \delta_k (x - x_k)$$

$\uparrow$ some slope

② Evolve

$$u(x, t_{\ell+1}) = u(x - at, t_\ell)$$



$$u_{k\,\ell+1} = \frac{1}{h_x} \int_{\Omega_k} u(x, t_{\ell+1})\, dx$$

$$= \frac{1}{h_x} \int_{x_{k-1/2}-at}^{x_{k-1/2}} u_{k-1\,\ell} + \delta_{k-1\,\ell}\,(x - x_{k-1})\, dx \quad (1)$$

$$+ \frac{1}{h_x} \int_{x_{k-1/2}}^{x_{k-1/2}+at} u_{k\,\ell} + \delta_{k\,\ell}\,(x - x_k) \quad (2)$$

Algorithm 6.1: non-limited linear reconstruction scheme for linear advection

Input : $\boldsymbol{u}_\ell$, grid function at time t_ℓ
ω, weight parameter in $[-1, 1]$
a, advection speed
h_t, h_x, grid spacing
Ω, list of finite-volume cells

Output : $\boldsymbol{u}_{\ell+1}$, grid function at time $t_{\ell+1}$

for each *cell* Ω_k

$$\delta_{k,\ell} = \cancel{\tfrac{1}{2}(1+\omega)\Delta^- u_{k,\ell} + \tfrac{1}{2}(1-\omega)\Delta^+ u_{k,\ell}} \qquad \{\text{compute slope}\}$$

$$u_{k,\ell+\frac{1}{2}} = u_{k,\ell} - \frac{h_t}{2} a \frac{\delta_{k,\ell}}{h_x} \qquad \{\text{compute cell value at intermediate time } t_{\ell+\frac{1}{2}}\}$$

$$\left.\begin{aligned} u^-_{k+\frac{1}{2}} &= u_{k,\ell+\frac{1}{2}} + \frac{\delta_{k,\ell}}{2} \\ u^+_{k-\frac{1}{2}} &= u_{k,\ell+\frac{1}{2}} - \frac{\delta_{k,\ell}}{2} \end{aligned}\right\} \qquad \{\text{compute linear reconstructions at cell interfaces}\}$$

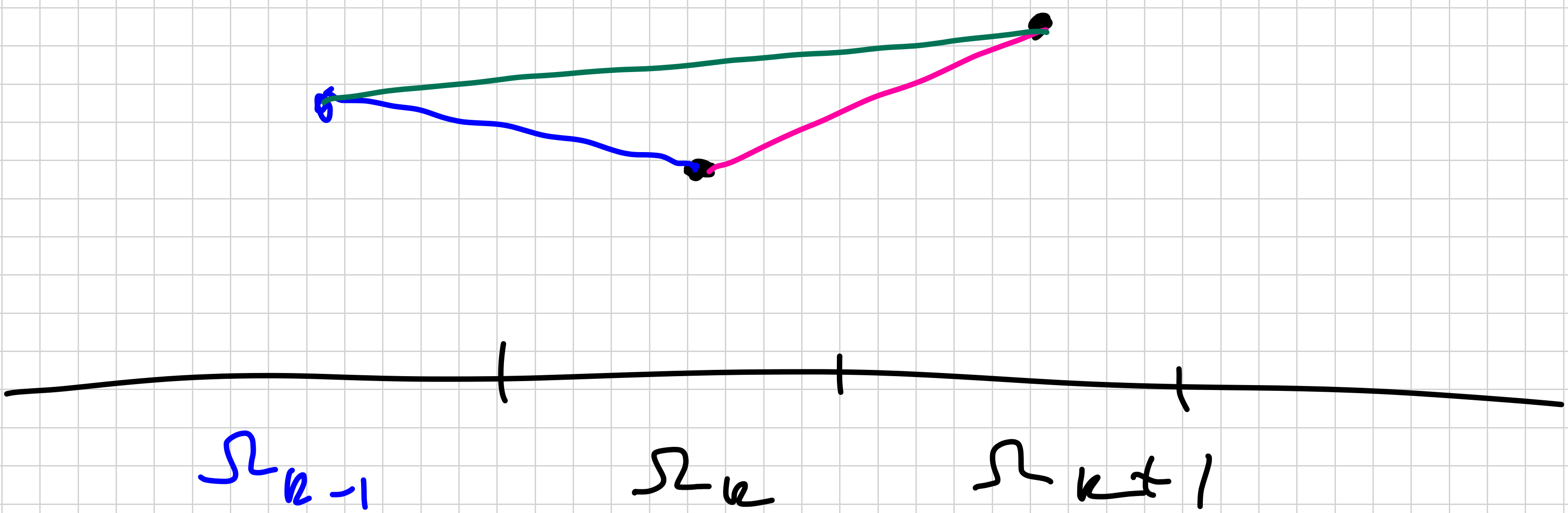
$$u_{k,\ell+1} = u_{k,\ell} - \frac{h_t}{h_x}\left(a u^-_{k+\frac{1}{2}} - a u^+_{k-\frac{1}{2}}\right) \qquad \{\text{use upwind flux } f^* = au\}$$

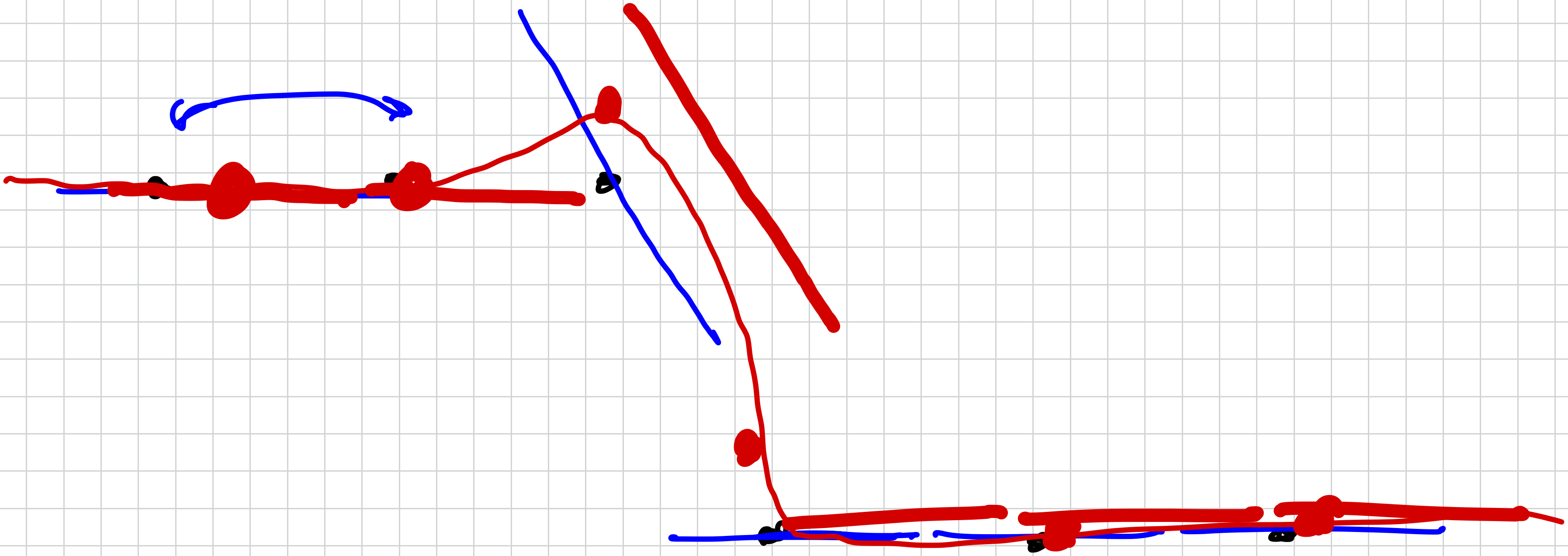
$\sigma = 0$ constant or "Godunov"

$\sigma = \dfrac{u_{k+1} - u_k}{h_x}$ Lax-Wendroff

$\sigma = \dfrac{u_k - u_{k-1}}{h_x}$ Beam-Warming

$\sigma = \dfrac{u_{k+1} - u_{k-1}}{2h_x}$ Fromm





need : limit slopes

look at ratio of slopes:

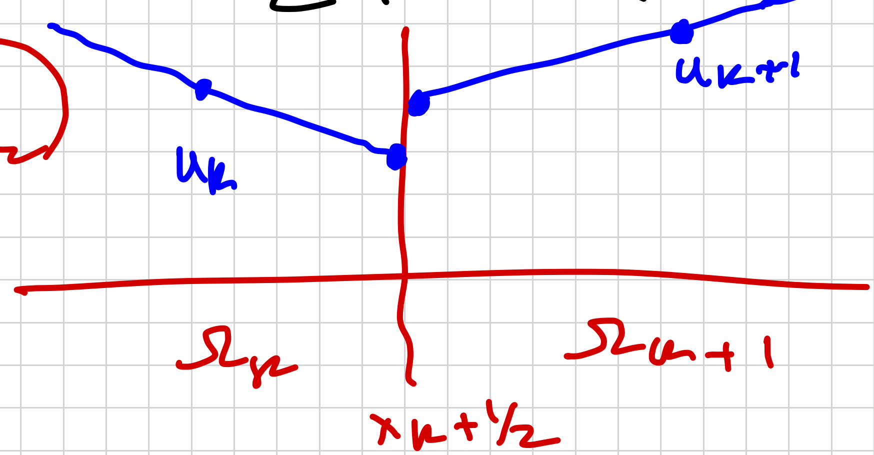
$$r_k = \frac{u_k - u_{k-1}}{u_{k+1} - u_k}$$

Seek $\phi(\,)$ so that

$$u^-_{k+1/2} = u_k + \frac{1}{2}\phi(r_k)(u_{k+1} - u_k)$$

$$u^+_{k+1/2} = u_{k+1} + \frac{1}{2}\phi(r_k)(u_{k+2} - u_{k+1})$$

$f^*(u^-_{k+1/2}, u^+_{k+1/2})$



New measure: total variation

consider $u(x)$

total variation in u = $TV(u(x), \Omega)$

$$= \int_{\Omega} \left| \frac{\partial u}{\partial x} \right| dx$$

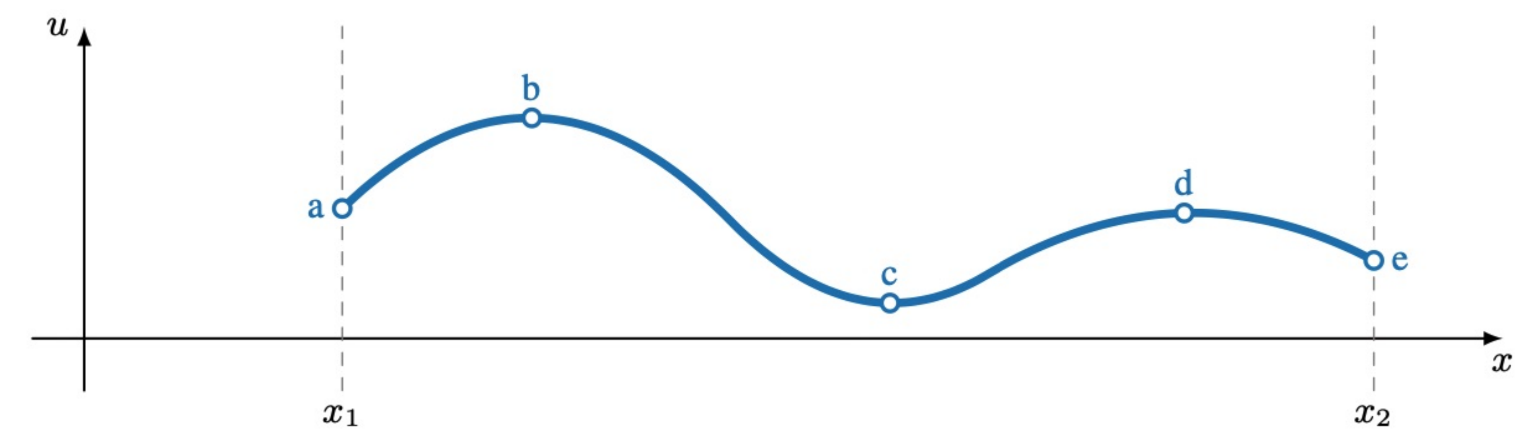
We say Total Variation Diminishing if

$$\frac{d}{dt} \int_{x_1(t)}^{x_2(t)} \left| \frac{\partial u(x,t)}{\partial x} \right| dx \leq 0$$

$x_1(t)$ and $x_2(t)$ are characteristics of

$$u_t + (f(u))_x = 0$$

①



Smooth function $u(x)$

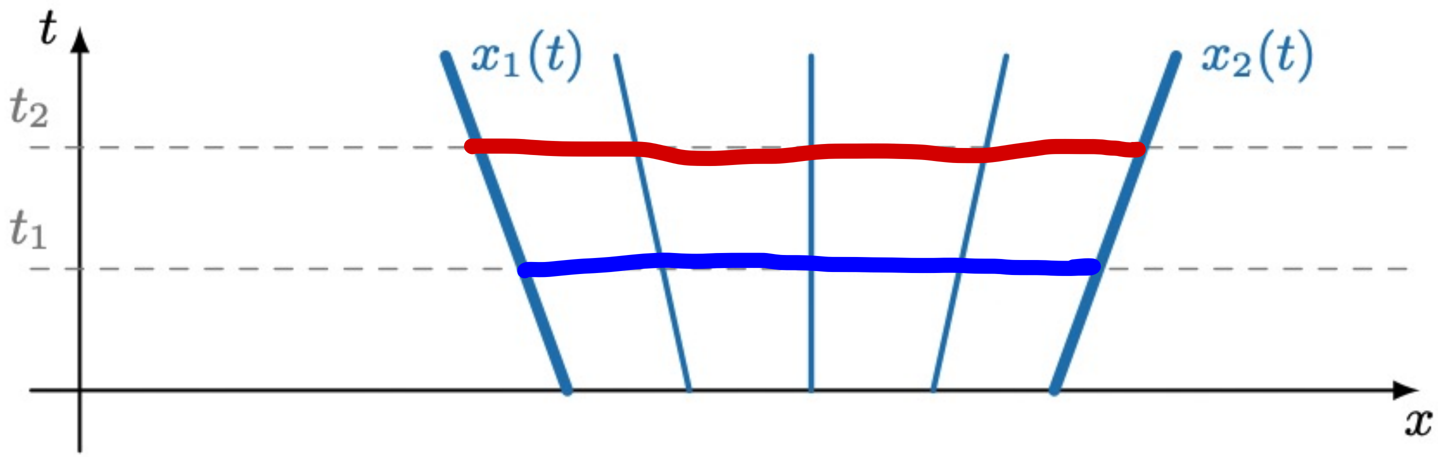
Then the TV is given by the extrema: $TV(u) = |u(x_b) - u(x_a)|$

$$+ |u(x_c) - u(x_b)|$$

$$+ |u(x_d) - u(x_c)|$$

$$+ |u(x_e) - u(x_d)|$$

2



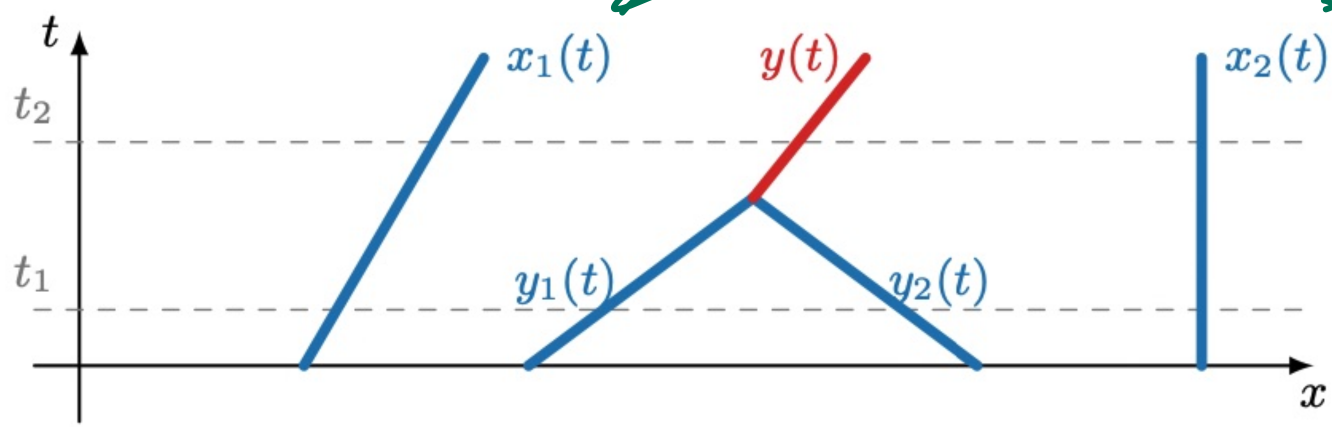
→ no new extrema between x_1 + x_2

→ $TV(u(x,t_1), \Omega(t_1)) = TV(u(x,t_2), \Omega(t_2))$

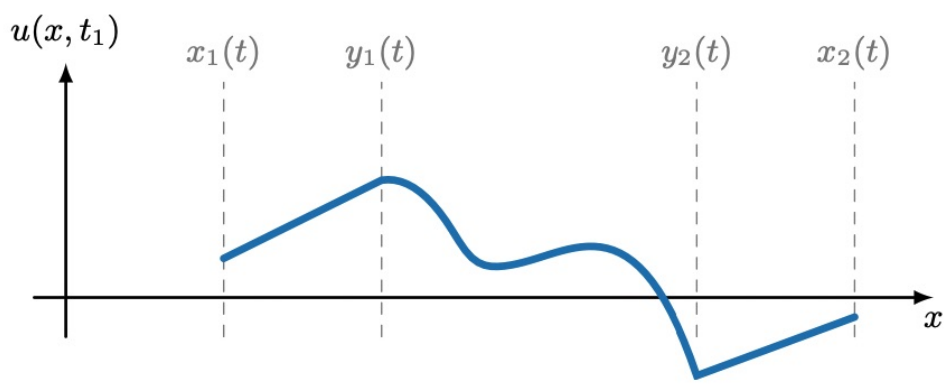
$$\frac{dTV}{dt} = 0$$

③

char.

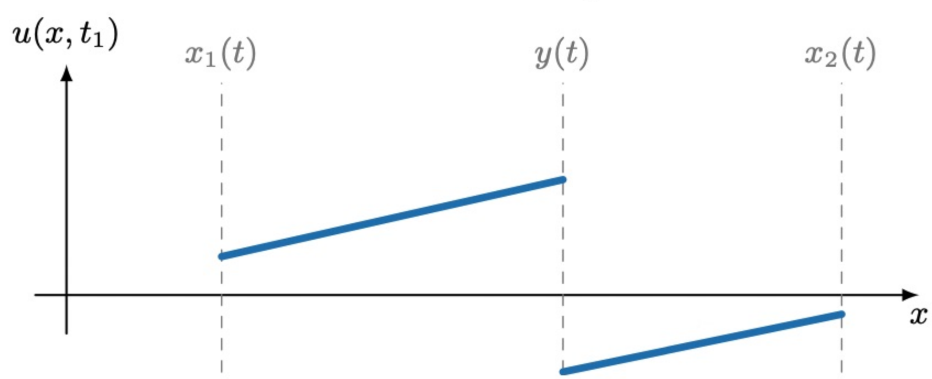


at t_1 =



$\Rightarrow \frac{dTV}{dt} < 0$

at t_2



Discrete analog:

let $\underline{u}_\ell = \begin{bmatrix} u_{1,\ell} \\ u_{2,\ell} \\ \vdots \\ u_{N,\ell} \end{bmatrix}$, and periodic

$$\rightarrow \boxed{TV(\underline{u}_\ell) = \sum_{k=1}^{N} \left| u_{k\ell} - u_{k-1,\ell} \right|}$$

$\Rightarrow$ TVD if

$$TV(\underline{u}_{\ell+1}) \leq TV(\underline{u}_\ell)$$

Definition 6.15: Linear Scheme

A numerical scheme

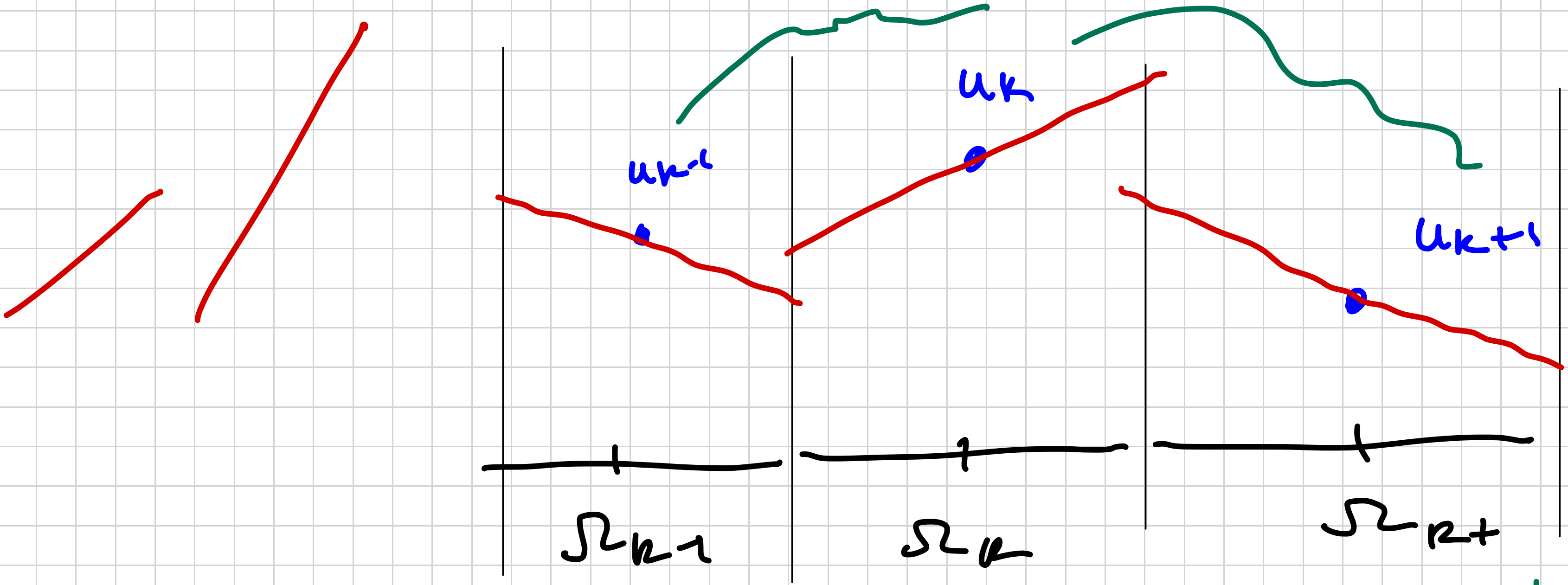
$$u_{k,\ell+1} = \sum_j c_j u_{k-j,\ell} \tag{6.116}$$

applied to the conservation law in Equation (6.2) with linear flux function $f(u) = au$ is called a linear scheme if all coefficients c_j are constant — i.e., they do not depend on the approximation $\mathbf{u}_\ell$ at time t_ℓ. Otherwise, the scheme is called nonlinear.

Godunov's theorem (which we state without proof) establishes that linear schemes of order higher than one cannot be TVD:

Theorem 6.16: Godunov's Theorem

Linear TVD schemes are at most first-order accurate.



$$r_k = \frac{u_k - u_{k-1}}{u_{k+1} - u_k}$$

$$u^-_{k+1/2} = u_k + \frac{\phi(r_k)}{2}(u_{k+1} - u_k)$$

what do want for $\phi(\cdot)$?

if $\mathrm{sign}(u_k - u_{k-1}) = \mathrm{sign}(u_{k+1} - u_k)$

then $\phi(r_k)\cdot(u_{k+1} - u_k)$

$= \min(u_k - u_{k-1}, u_{k+1} - u_k)$

$\Rightarrow \phi(r_k) = \min(r_k, 1)$

if $\mathrm{sign}(u_k - u_{k-1}) \neq \mathrm{sign}(u_{k+1} - u_k)$

then want $\phi(u_{k+1} - u_k) = 0$

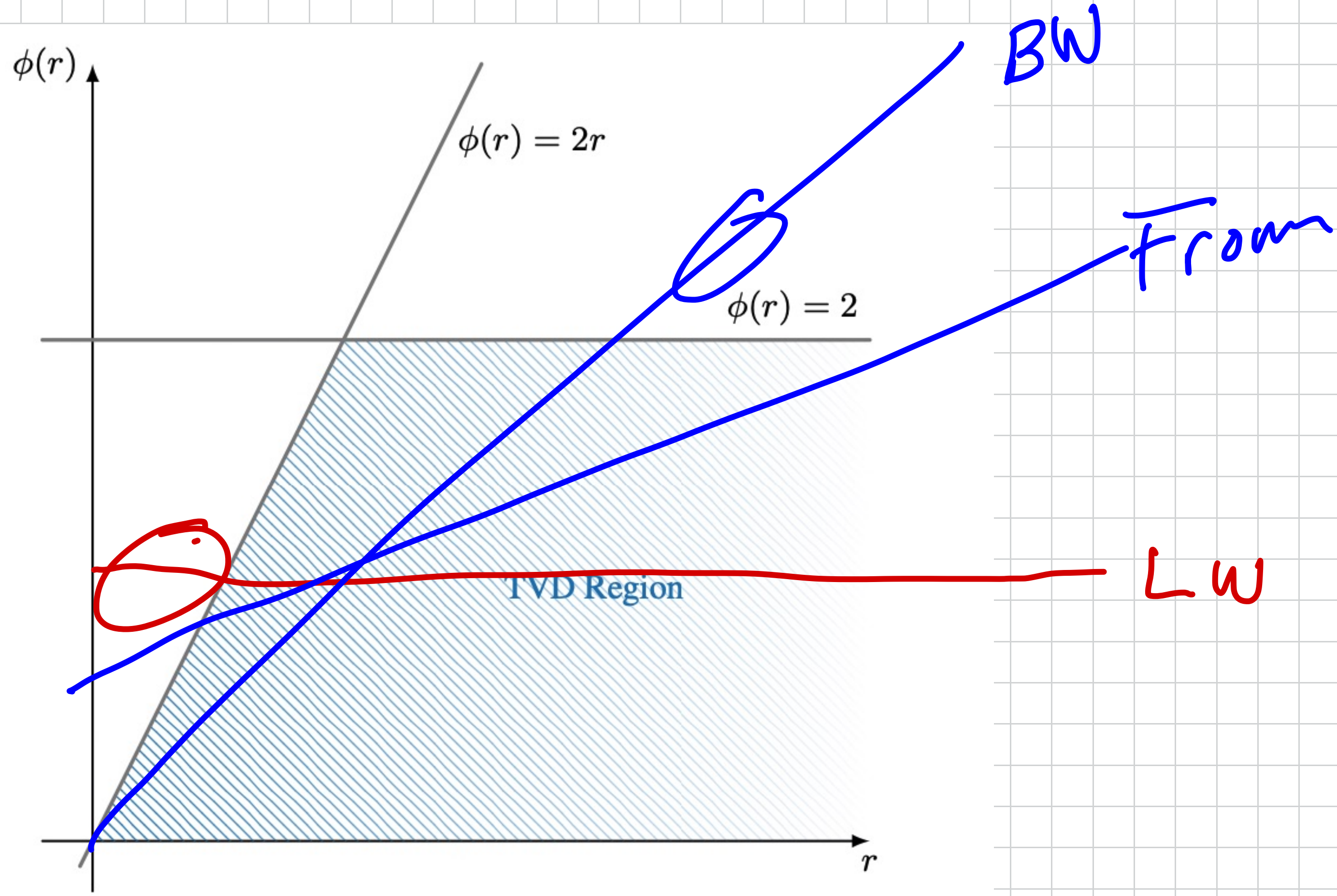
$$\phi(r) = \max(0, \min(r, 1))$$

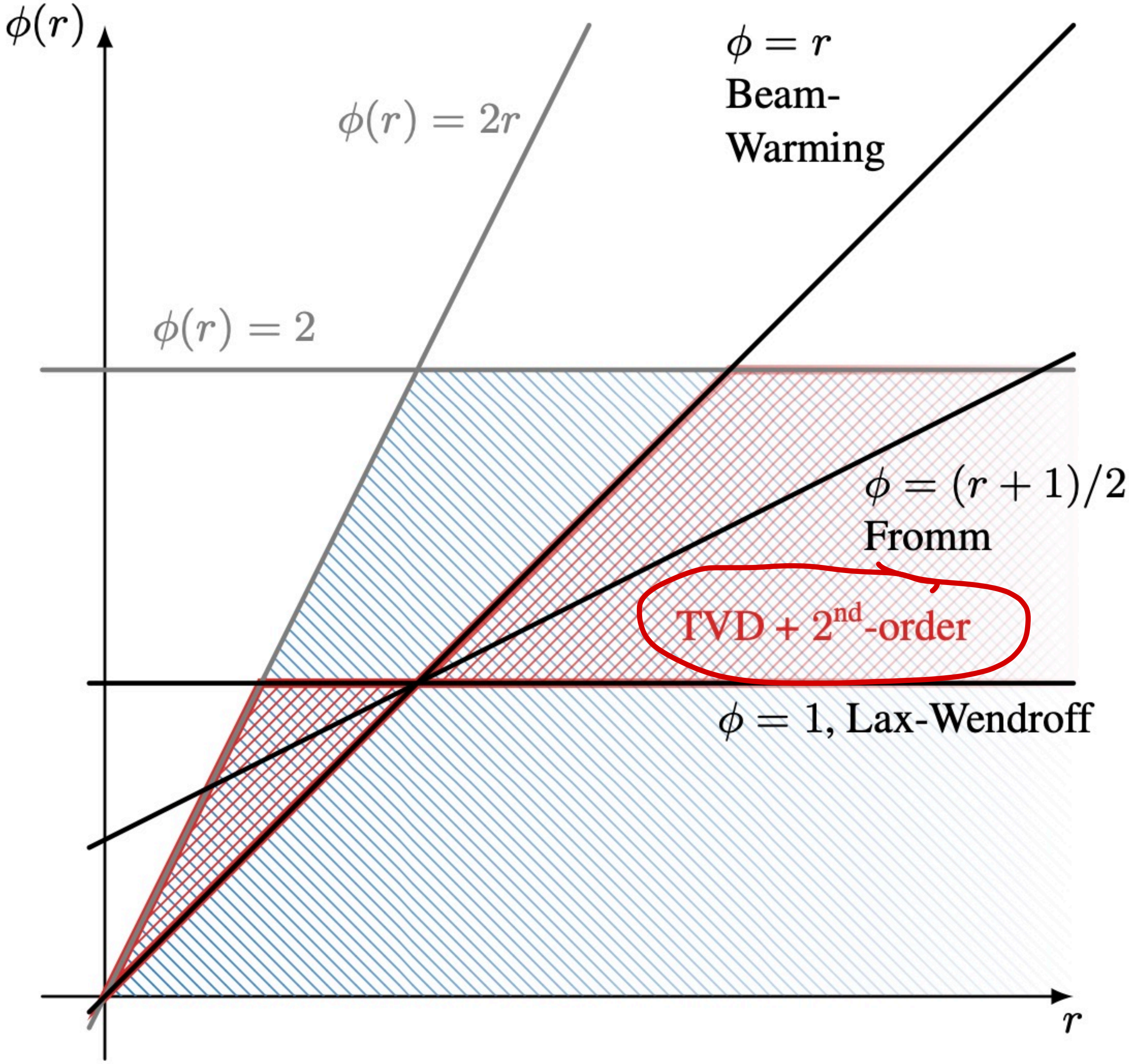
what conditions on $\phi(\,)$ give TVD?

$0 \le \phi \le 2$

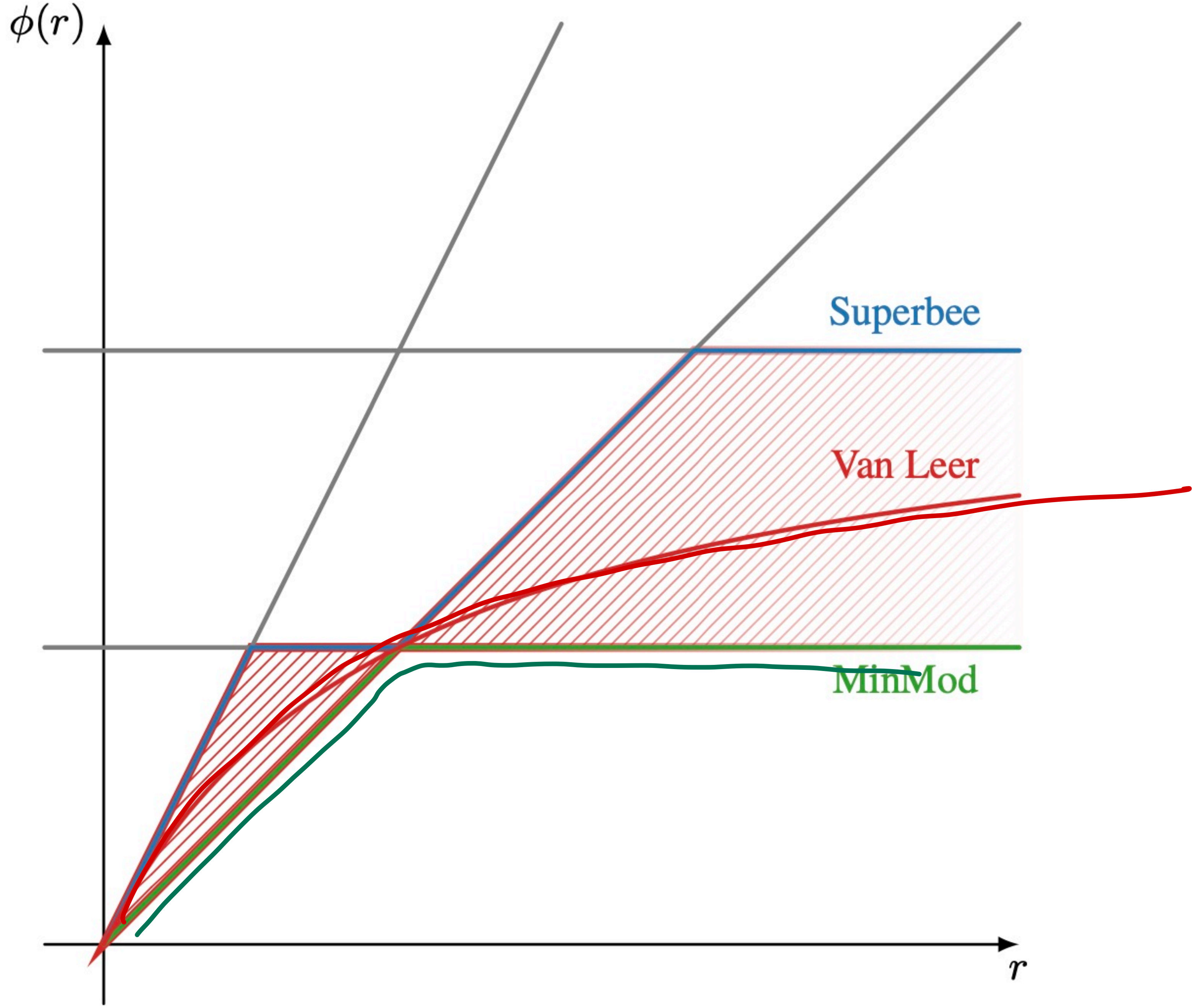
and

$0 \le \phi \le 2r$





$\phi(r)$
$\phi = r$
Beam-
Warming
$\phi(r) = 2r$
$\phi(r) = 2$
$\phi = (r+1)/2$
Fromm
TVD + 2nd-order
$\phi = 1$, Lax-Wendroff
r


$\phi(r)$
Superbee
Van Leer
MinMod
r