Goals:

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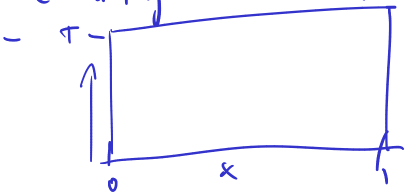
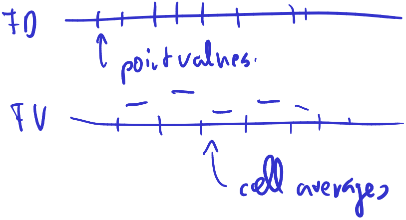
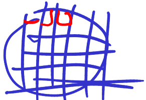
$\partial_t u + \partial_x u = 0$

- Causality identifies time variables

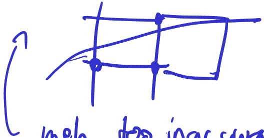




FD:

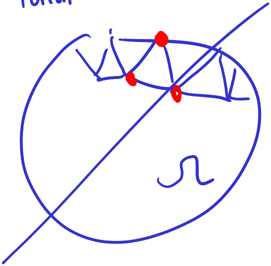


point values



meh, too inaccurate

FE Idea:



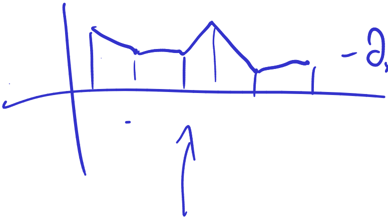
use cell wise linear functions based on vertex point value DOFs

$-\Delta u = f$ ← Poisson "BVP"

$-(\partial_{xx}^2 + \partial_{yy}^2)u = f$

to make solvable: $u = g|_{\partial\Omega}$

$-\partial_{xx}^2 u = f$



multiply by test fn $-\Delta u \varphi = f \varphi$ $(\varphi \in C^\infty)$

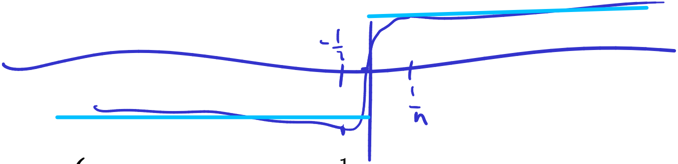
$-\int \Delta u \varphi = \int f \varphi$

weakform — IBP → $\int \nabla u \cdot \nabla \varphi = \int f \varphi$

"weak derivative"

Sobolev spaces

Function Spaces



Consider

$$f_n(x) = \begin{cases} -1 & x \leq -\frac{1}{n}, \\ \frac{3n}{2}x - \frac{n^3}{2}x^3 & -\frac{1}{n} < x < \frac{1}{n}, \\ 1 & x \geq 1/n. \end{cases}$$

Converges to the step function. Problem?

$f_n \in C^1(\mathbb{R})$ but f is not even cont.

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Norms

Definition (Norm)

A norm $\|\cdot\|$ maps an element of a *vector space* into $[0,\infty)$. It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$ ← definiteness
- $\|\lambda x\| = |\lambda|\|x\|$
- $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

Convergence

Definition (Convergent Sequence)

$x_n \to x :\Leftrightarrow \|x_n - x\| \to 0$ (convergence in norm)

Definition (Cauchy Sequence)

for all $\varepsilon > 0$ there exists an n for which

$\|x_\nu - x_\mu\| \leq \varepsilon$ for $\nu, \mu \geq n$,

Banach Spaces

( )

Definition (Complete/"Banach" space)

Cauchy ⇒ Convergent

What's special about Cauchy sequences?

Limit (in same function space) shows up out of thin air.

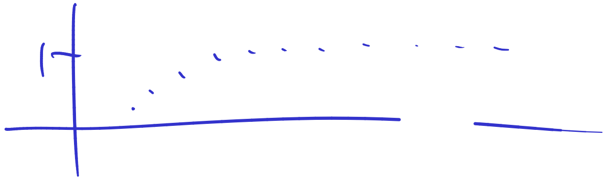
Counterexamples?

$(\mathbb{Q}, |\cdot|)$

$(C^1, \|\cdot\|_{L^1})$

$\|f\|_{L^1} = \int_\Omega |f|\,dx$

sup v. max:



More on C^0

Let $\Omega \subseteq \mathbb{R}^n$ be open. Is $C^0(\Omega)$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

(0, 1) $f(x) = \frac{1}{x}$

Problem: $\|f\|_\infty$ not defined , $(C^0(\Omega), \|\cdot\|_\infty)$ not Banach for Ω open

Is $C^0(\bar{\Omega})$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

↑ closed

Assume (f_i) Cauchy w/ sup norm.

- Let $x \in \bar{\Omega}$. $(f_i(x))_{i\in\mathbb{N}} \leftarrow$ Cauchy sequence in $(\mathbb{R}, |\cdot|)$.

$\Rightarrow$ ($\mathbb{R}$ complete) there exists $a \in \mathbb{R}$ so that $f(x_i) \to \xi$ $(i\to\infty)$.

Assemble candidate limit func. Found of pointwise limits

Let $\varepsilon > 0$, Then exists an $N \in \mathbb{N}$ so that

$$\sup_{x \in \bar{\Omega}} |f_n(x) - f_m(x)| < \varepsilon \quad \text{for all } n, m \geq N.$$

Take the limit $m \to \infty$:

$$\max_{x \in \bar{\Omega}} |f_n(x) - f(x)| < \varepsilon \quad \Rightarrow \quad \|f_n - f\|_\infty \to 0.$$

↑ uniform convergence

C^m Spaces

Let $\Omega \subseteq \mathbb{R}^n$.

$f : \Omega \to \mathbb{R}$

Consider a multi-index $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ and define the symbols

$$D^{\vec{k}} f = \frac{\partial^{|\vec{k}|}}{\partial^{k_1}_{x_1} \cdots \partial^{k_n}_{x_n}} f \qquad |\vec{k}| = k_1 + \cdots + k_n$$

Definition (C^m Spaces)

$$C^m(\Omega) = \{ f \in C^0(\Omega) : D^{\vec{k}} f \in C^0(\Omega) \text{ s.t. } |\vec{k}| \le m \}$$

$$C^\infty(\Omega) = \{ f \in C^0(\Omega) : D^{\vec{k}} f \in C^0(\Omega) \text{ for all } \vec{k} \}$$

"0 at bdry" $\rightarrow$ $C_0^m(\Omega) = \{ f \in C^m(\Omega) : f \text{ have compact support} \}$

E.g. C^2: $\partial_{xx}^2 \partial_{yy}^2 u \in C^0$? no!

$$\left.\begin{matrix} \partial_{xx}^2 u \\ \partial_x \partial_y u \\ \partial_{yy}^2 u \end{matrix}\right\} \in C^0$$ yes!

"support" of a function: $\{x \in \mathbb{R} : f(x) \neq 0\}$

"compact": closed + bounded (<u>only</u> in $\mathbb{R}^n$)

L^p Spaces

Let $1 \leq p < \infty$.

Definition (L^p Spaces)

$$L^p(\Omega) := \left\{ u : (u : \mathbb{R} \to \mathbb{R}) \text{ measurable}, \int_\Omega |u|^p \, dx < \infty \right\},$$

$$\|u\|_p := \left(\int_\Omega |u|^p \, dx \right)^{1/p} .$$

Definition (L^∞ Space)

$$L^\infty(\Omega) := \{ u : (u : \mathbb{R} \to \mathbb{R}), |u(x)| < \infty \text{ almost everywhere} \},$$

$$\|u\|_\infty = \inf \{ C : |u(x)| \leq C \text{ almost everywhere} \} .$$

L^p Spaces: Properties

Theorem (Hölder's Inequality)

For $1 \leq p, q \leq \infty$ *with* $1/p + 1/q = 1$ *and measurable* u *and* v,

$$\|uv\|_1 \leq \|u\|_p \|v\|_q$$

(gen. of Cauchy-Schwarz)

Theorem (Minkowski's Inequality (Triangle inequality in L^p))

For $1 \leq p \leq \infty$ *and* $u, v \in L^p(\Omega)$,

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p$$

Inner Product Spaces

Let V be a vector space.

Definition (Inner Product)

An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that for any $f, g, h \in V$ and $\alpha \in \mathbb{R}$

$$\begin{aligned} \langle f, f \rangle &\geq 0, \\ \langle f, f \rangle &= 0 \Leftrightarrow f = 0, \\ \langle f, g \rangle &= \langle f, g \rangle, \\ \langle \alpha f + g, h \rangle &= \alpha \langle f, h \rangle + \langle g, h \rangle . \end{aligned}$$

Definition (Induced Norm)

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

Hilbert Spaces

Definition (Hilbert Space)

An inner product space that is complete under the induced norm.

Let Ω be open.

Theorem (L^2)

$L^2(\Omega)$ equals the closure of (set of all limits of Cauchy sequences in) $C_0^\infty(\Omega)$ under the induced norm $\|\cdot\|_2$.

Theorem (Hilbert Projection (e.g. Yosida '95, Thm. III.1))

Weak Derivatives

Define the space L^1_{loc} of locally integrable functions.

Definition (Weak Derivative)

$v \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of $u \in L^1_{\text{loc}}(\Omega)$ of multi-index order $\boldsymbol{k}$ if

Weak Derivatives: Examples (1/2)

Consider all these on the interval $[-1, 1]$.

$$f_1(x) = 4(1 - x)x$$

$$f_2(x) = \begin{cases} 2x & x \leq 1/2, \\ 2 - 2x & x > 1/2. \end{cases}$$

Weak Derivatives: Examples (2/2)

$$f_3(x) = \sqrt{\frac{1}{2}} - \sqrt{|x - 1/2|}$$

Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}_0$ and $1 \leq p < \infty$.

Definition ((k, p)-Sobolev Norm/Space)

More Sobolev Spaces

$W^{0,2}$?

$W^{s,2}$?

$H_0^1(\Omega)$?

Outline

An Elliptic Model Problem

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^1(\Omega)$.

$$\begin{aligned} -\nabla \cdot \nabla u + u &= f(x) \quad (x \in \Omega), \\ u(x) &= 0 \qquad (x \in \partial\Omega). \end{aligned}$$

Let $V := H_0^1(\Omega)$. Integration by parts? (Gauss's theorem applied to $a\boldsymbol{b}$):

Weak form?

Motivation: Bilinear Forms and Functionals

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv = \int fv.$$

This is the weak form of the strong-form problem. The task is to find a $u \in V$ that satisfies this for all test functions $v \in V$.

Recast this in terms of bilinear forms and functionals:

Dual Spaces and Functionals

Bounded Linear Functional

Let $(V, \|\cdot\|)$ be a Banach space. A linear functional is a linear function $g : V \to \mathbb{R}$. It is bounded ($\Leftrightarrow$ continuous) if there exists a constant C so that $|g(v)| \leq C \|v\|$ for all $v \in V$.

Dual Space

Let $(V, \|\cdot\|)$ be a Banach space. Then the dual space V' is the space of bounded linear functionals on V.

Dual Space is Banach (cf. e.g. Yosida '95 Thm. IV.7.1)

V' is a Banach space with the dual norm

Functionals in the Model Problem

Is g from the model problem a bounded functional? (In what space?)

That bound felt loose and wasteful. Can we do better?

Riesz Representation Theorem (1/3)

Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Theorem (Riesz)

Let g be a bounded linear functional on V, i.e. $g \in V'$. Then there exists a unique $u \in V$ so that $g(v) = \langle u, v \rangle$ for all $v \in V$.

Riesz Representation Theorem: Proof (2/3)

Have $w \in N(g)^{\perp} \setminus \{0\}$, $\alpha = g(w) \neq 0$, and $z := v - (g(v)/\alpha)w \perp w$.

Riesz Representation Theorem: Proof (3/3)

Uniqueness of u?

Back to the Model Problem

$$
\begin{aligned}
a(u,v) &= \langle \nabla u, \nabla v \rangle_{L^2} + \langle u, v \rangle_{L^2} \\
g(v) &= \langle f, v \rangle_{L^2} \\
a(u,v) &= g(v)
\end{aligned}
$$

Have we learned anything about the solvability of this problem?

Poisson

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^{-1}(\Omega)$.

This is called the Poisson problem (with Dirichlet BCs).

Weak form?

Ellipticity

Let V be Hilbert space.

V-Ellipticity

A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called coercive if there exists a constant $c_0 > 0$ so that

and a is called continuous if there exists a constant $c_1 > 0$ so that

If a is both coercive and continuous on V, then a is said to be V-elliptic.

Lax-Milgram Theorem

Let V be Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Lax-Milgram, Symmetric Case

Let a be a V-elliptic bilinear form that is also symmetric, and let g be a bounded linear functional on V.
Then there exists a unique $u \in V$ so that $a(u, v) = g(v)$ for all $v \in V$.

Back to Poisson

Can we declare victory for Poisson?

Can this inequality hold in general, without further assumptions?

Poincaré-Friedrichs Inequality (1/3)

Theorem (Poincaré-Friedrichs Inequality)

Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $u \in H_0^1(\Omega)$. Then there exists a constant $C > 0$ such that

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}.$$

Poincaré-Friedrichs Inequality (2/3)

Prove the result in $C_0^\infty(\Omega)$.

Poincaré-Friedrichs Inequality (3/3)

Prove the result in $H_0^1(\Omega)$.

Back to Poisson, Again

Show that the Poisson bilinear form is coercive.

Draw a conclusion on Poisson:

Outline

Ritz-Galerkin

Some key goals for this section:

- How do we use the weak form to compute an approximate solution?
- What can we know about the accuracy of the approximate solution?

Can we pick one underlying principle for the construction of the approximation?

Galerkin Orthogonality

$$a(u, v) = g(v) \qquad \text{for all } v \in V, a(u_h, v_h) \quad = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Observations?

Céa's Lemma

Let $V \subset H$ be a closed subspace of a Hilbert space H.

Céa's Lemma

Let $a(\cdot, \cdot)$ be a coercive and continuous bilinear form on V. In addition, for a bounded linear functional g on V, let $u \in V$ satisfy

$$a(u, v) = g(v) \qquad \text{for all } v \in V.$$

Consider the finite-dimensional subspace $V_h \subset V$ and $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Then

Céa's Lemma: Proof

Recall Galerkin orthgonality: $a(u_h - u, v_h) = 0$ for all $v_h \in V_h$. Show the result.

Elliptic Regularity

Definition (H^s Regularity)

Let $m \geq 1$, $H_0^m(\Omega) \subseteq V \subseteq H^m(\Omega)$ and $a(\cdot,\cdot)$ a V-elliptic bilinear form. The bilinear form $a(u,v) = \langle f, v\rangle$ for all $v \in V$ is called H^s regular, if for every $f \in H^{s-2m}$ there exists a solution $u \in H^s(\Omega)$ and we have with a constant $C(\Omega, a, s)$,

Theorem (Elliptic Regularity (cf. Braess Thm. 7.2))

Let a be a H_0^1-elliptic bilinear form with sufficiently smooth coefficient functions.

Elliptic Regularity: Counterexamples

Are the conditions on the boundary essential for elliptic regularity?

Are there any particular concerns for mixed boundary conditions?

Estimating the Error in the Energy Norm

Come up with an idea of a bound on $\|u - u_h\|_{H^1}$.

What's still to do?

L^2 Estimates

Let H be a Hilbert space with the norm $\|\cdot\|_H$ and the inner product $\langle \cdot, \cdot \rangle$. (Think: $H = L^2$, $V = H^1$.)

Theorem (Aubin-Nitsche)

Let $V \subseteq H$ be a subspace that becomes a Hilbert space under the norm $\|\cdot\|_V$. Let the embedding $V \to H$ be continuous. Then we have for the finite element solution $u \in V_h \subset V$:

if with every $g \in H$ we associate the unique (weak) solution φ_g of the equation (also called the dual problem)