Today 2/21

1. Go back to linear algebra
2. State problem as a "minimization"
3. Develop a "weak form" of the problem.

Goal: find $u(x)$ s.t. $-u_{xx} = f$ $(0,1)$
such that
$u(0) = 0$
$u'(1) = 1$

Easier: find $\underline{x}$ s.t. $A\underline{x} = \underline{b}$
$\underline{x} \in \mathbb{R}^n$
$b \in \mathbb{R}^n$
square, non-singular

$\rightarrow$ find $\underline{x}$ s.t.
$\underline{b} - A\underline{x} = 0$

$\rightarrow$ find $\underline{x}$ s.t.
$\underline{v}^T(b - Ax) = 0 \quad \forall\ \underline{v} \in \mathbb{R}^n$

$\rightarrow$ let $\{u_i\}_{i=1}^n$ be a basis for $\mathbb{R}^n$
find x s.t.
$u_i^T(b - Ax) = 0 \quad \forall\ u_i$

$\| b - Ax \|_2 \to \min.$

"least squares"

$\longrightarrow x$ s.t. $A^T A x = A^T b \Rightarrow A^T(b - Ax) = 0$

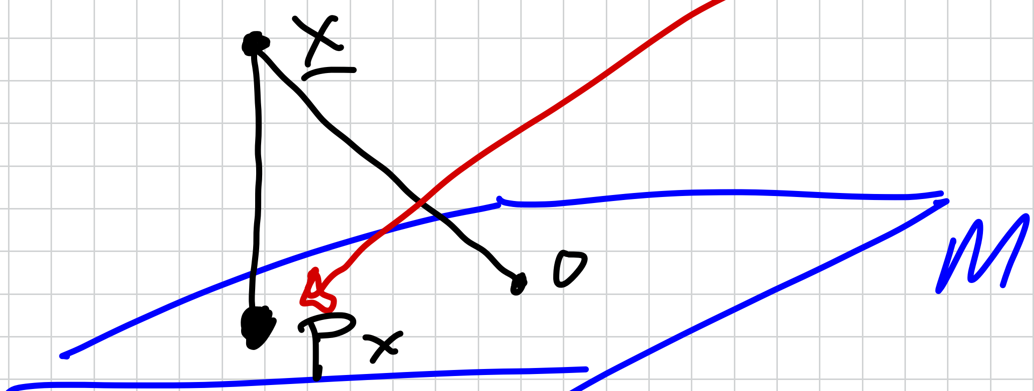
Another question:

let $x \in \mathbb{R}^n$

let $M \subset \mathbb{R}^n$ subspace, m-dim.

let $v_1 \cdots v_m$ span M (basis for M)

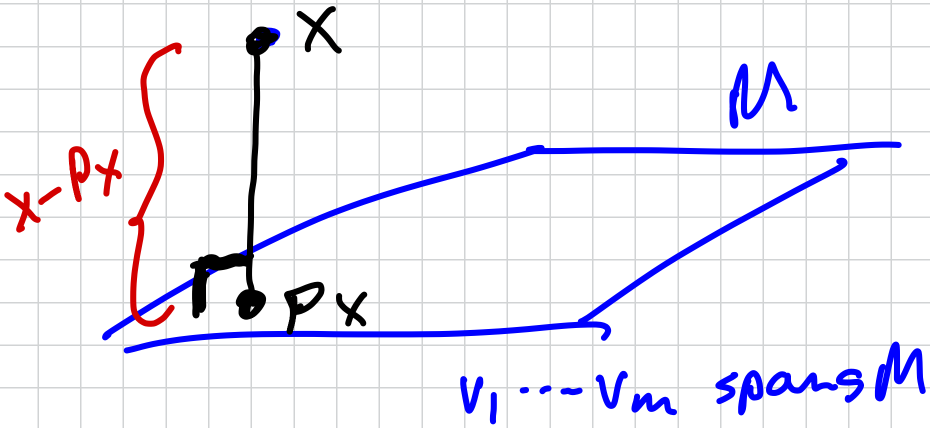
what is the closest thing in M to x?



P = projector

$\rightarrow$ $P^2 = P$, $(I-P)^2 = I - 2P + P^2$

$= I - P$



Px m dofs.

mxn nx1

want $x - Px \perp M$ m constraints

$$Px = Vy = \text{"linear comb of } v_i\text{'s"}$$

$$= \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = y_1 \underline{v}_1 + y_2 \underline{v}_2 + \cdots + y_m \underline{v}_m$$

need $x - Px \perp M$

or

$$x - Px \perp v_1\ v_2 \cdots v_m$$

$$x - Px \perp v_1 \cdots v_m$$

$$\hookrightarrow (x - Px, v_i) = 0 \quad \forall v_i$$

$$\hookrightarrow (x - Vy, v_i) = 0 \quad \forall v_i$$

$$\hookrightarrow V^T (x - Vy) = 0$$

$$\hookrightarrow V^T V y = \underbrace{V^T x}_{n \times m}$$

$$\to y = (V^T V)^{-1} V^T x$$

$$\to Vy = \underbrace{V (V^T V)^{-1} V^T}_{P} x$$

Theorem

Let P be the orthogonal projector onto M:

for any x

$$\min_{y \in M} \|x - y\|_2 = \|x - Px\|_2$$

New problem: given $f(x)$

find $p_n(x)$ s.t. $p_n \sim f(x)$ "close to"

① max norm:

$$\|f\|_\infty = \max_{a \le x \le b} |f(x)|$$

on $C([a,b])$

all continuous fcns.

② 2-norm

$$\|f\|_2 = \int_a^b |f(x)|^2 dx$$

on $C([a,b])$

let $P_k = \{ p(x) \mid p(x) =$ degree-k or less polynomial $\}$

min-max: find $p_k^*(x) = \underset{p_k \in P_k}{\operatorname{argmin}} \| f - p_k(x) \|_\infty$

least-squares: find $p_k^*(x) = \underset{p_k \in P_k}{\operatorname{argmin}} \| f - p_k(x) \|_2$

want p_k s.t.

$\| f - p_k \|_2$ is minimized.

fixed

How?

Any p_k can be written

$$= \sum_{i=0}^{k} a_i \, \phi_i(x)$$

for some basis $\{\phi_i\}_0^k$

Basis examples

$\phi_i = 1, x, x^2, x^3 \ldots$

= monomials.

$\phi_i = 1, x, \frac{3x^2-1}{2}, \frac{5x^3-x}{2}, \ldots$

= Legendre

$$\phi_i = \frac{\prod (x - x_j)}{\prod (x_i - x_j)}$$ Lagrange

minimize $F = \| f - p_k(x) \|_2$ on $[0,1]$

$$= \int_0^1 \left(f(x) - p_k(x) \right)^2 dx$$

$$= \int_0^1 \left(f(x) - \sum_{i=0}^{k} a_i \phi_i \right)^2 dx$$

$$= \int_0^1 f^2 dx - 2 \sum_{i=0}^{k} a_i \int_0^1 f \phi_i \, dx + \sum_{i=0}^{k} \sum_{j=0}^{k} a_i a_j \int \phi_i \phi_j \, dx$$

$$\frac{\partial F}{\partial a_i} = 0 - 2 \int_0^1 f \phi_i \, dx + \sum_{j=0}^{k} 2 a_j \phi_j \phi_i \, dx$$

$$= 0$$

$$\rightarrow \quad \sum_{j=0}^{k} \int a_j \phi_j \phi_i \, dx = \int_0^1 f \phi_i \, dx \quad \forall i$$

$$\langle \phi_j, \phi_i \rangle = \int_0^1 \phi_j \phi_i$$

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & \cdots & \langle \phi_0, \phi_k \rangle \\ \langle \phi_1, \phi_0 \rangle & \cdots & \langle \phi_1, \phi_k \rangle \\ & & \vdots \\ & & \langle \phi_k, \phi_k \rangle \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \vdots \\ \langle f, \phi_k \rangle \end{bmatrix}$$

$$M \quad \underline{a} = \underline{\hat{f}}$$

need quadrature for $\int f \phi_i$

$\int \phi_j \phi_i$

$$\int f(x)\,dx \approx \sum_{i=0}^{n-1} w_i\, f(x_i)$$

nodes

quadrature weights

Goal: given $g(x)$

find $u(x)$ s.t.

$$u(x) = g(x)$$

Find $u(x) \in P_k$ s.t.

$$u(x) \approx g(x)$$

Find $u(x) \in P_k$ s.t.

$$\int_0^1 \left(u(x) - g(x)\right) v(x)\, dx \quad \forall\ v(x) \in P_k$$

$$\int_0^1 u(x)\, v(x)\, dx = \int_0^1 \underbrace{g(x)}_{\text{Known}}\, v(x)\, dx$$

let $u = \sum_{i=0}^{k} u_i \phi_i(x)$ ← basis

Find $\underline{u} = \{u_0 \cdots u_k\}$ s.t.

$$\int_0^1 \sum u_i \phi_i(x) \, v(x) \, dx = \int g(x) v(x) \, dx \quad \forall v(x)$$

$$\rightarrow \int_0^1 \sum u_i \underbrace{\phi_i \cdot \phi_j}_{\langle \phi_i, \phi_j \rangle}(x) \, dx = \int \underbrace{g(x) \phi_j(x)}_{\langle g, \phi_j \rangle} \, dx \quad \forall \phi_j$$

$$\rightarrow \begin{bmatrix} \langle \phi_0, \phi_0 \rangle & \cdots & \langle \phi_0, \phi_k \rangle \\ & & \\ & & \langle \phi_k, \phi_k \rangle \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} \langle g, \phi_0 \rangle \\ \vdots \\ \langle g, \phi_k \rangle \end{bmatrix}$$

find $u \in V$ s.t.

$$u = g(x)$$

weak form:

find $u \in V$ s.t.

$$\int u v \, dx = \int g v \, dx \quad \forall v \in V$$

original goal: find $u \in V$ s.t.

$$\begin{cases} -u_{xx} = f(x) \text{ on } [0,1] \\ u(0)=0 \\ u'(1)=0 \end{cases}$$

!!

Weak form of the problem:

find $u \in V$ s.t.

$$\int_0^1 -u_{xx} \cdot v \, dx = \int_0^1 f \cdot v \, dx$$

note: make sure V satisfies $v(0)=0 \;\forall v \in V$.

IBP

$$\int_0^1 u_x v_x \, dx - u_x v \Big|_0^1 = \int_0^1 f v \, dx$$

$= 0$

assume $u_x = 0$ at $x = 1$

Section 4.1

find $u \in V$ s.t.

$$\underbrace{\int_0^1 u_x v_x \, dx}_{\text{bilinear form}} = \underbrace{\int_0^1 f v \, dx}_{\text{linear functional}}$$

$a(u,v) : V \times V \to \mathbb{R}$

$\ell(v) : V \to \mathbb{R}$

function space

find $u \in V$ s.t.

$$a(u,v) = \ell(v) \quad \forall v \in V.$$

$$-u_{xx} = f$$

Attempt #1: define V = all continuous functions

look in a smaller subspace:

V^h = all continuous, piecewise linear functions on

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$