Today 3/18

Goal: F.E. Assembly in 2D

Model problem:

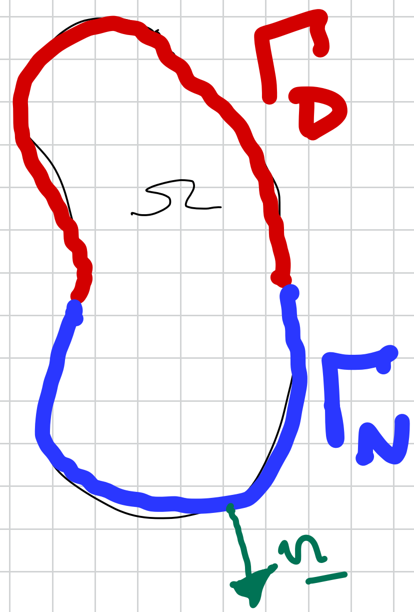
$$-\nabla\cdot k(x)\nabla u = f \text{ in } \Omega$$

$$u = g_D \text{ on } \Gamma_D$$

"Dirichlet"

$$n\cdot k(x)\nabla u = g_N \text{ on } \Gamma_N$$

Neumann



"∇·" "∇u"

$$-\begin{bmatrix}\partial_x & \partial_y\end{bmatrix}\cdot k(x)\begin{bmatrix}\partial_x\\ \partial_y\end{bmatrix}u$$

Case 1:

$k=1$

$g_N=0$

$\rightarrow n\cdot\nabla u = 0$

The weak form:

$$-\nabla \cdot K(x)\, \nabla u = f$$

$$\to \int_\Omega -\nabla \cdot K \nabla u \; v \, dx = \int_\Omega f v \, dx \qquad \forall v \in V$$

I.B.P.

$$\to \int_\Omega K \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} n \cdot K \nabla u \, v \, dx = \int_\Omega f v \, dx$$

let V_0 = space with $u(x) = 0$ for $x \in \Gamma_D$

$\to$ Find $u \in V_g$ s.t.

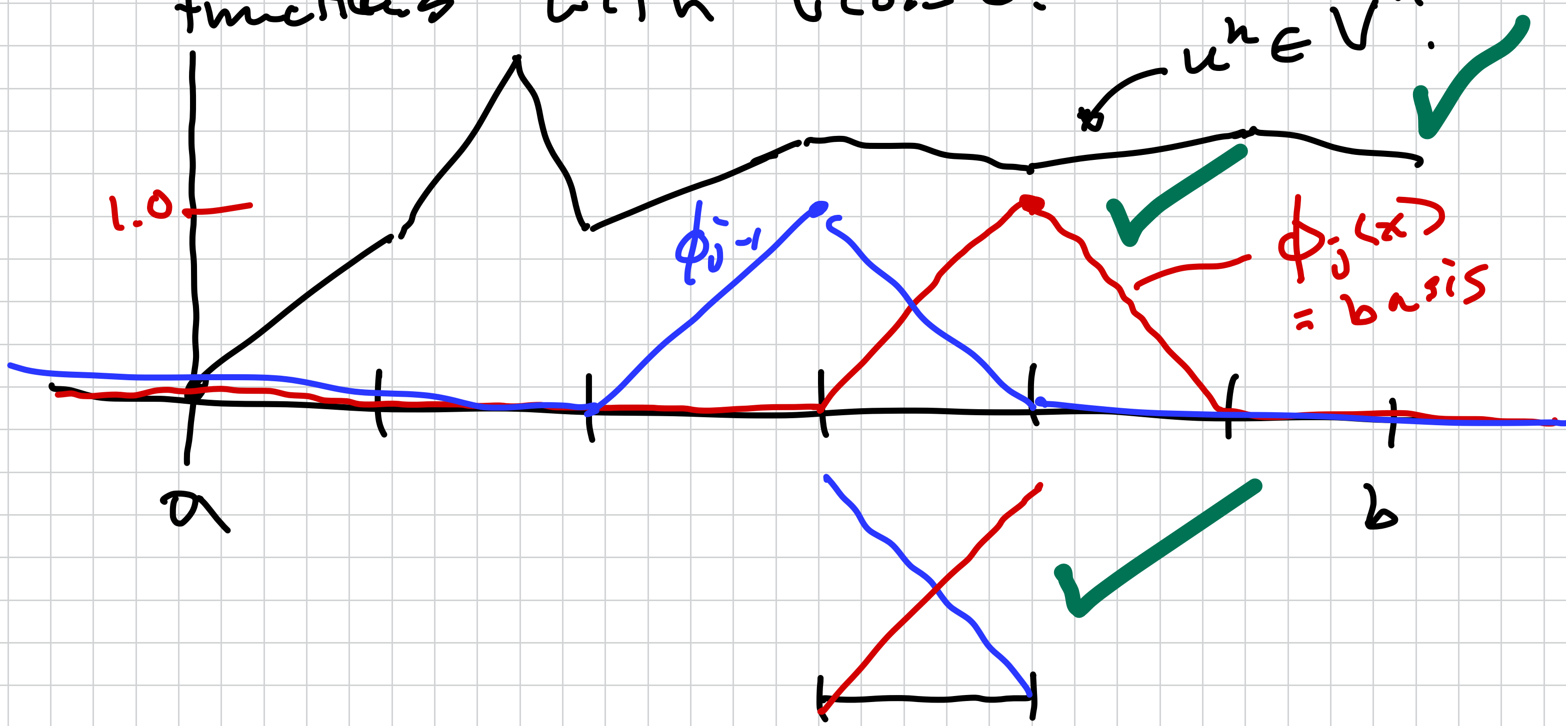
$$\int_\Omega K \nabla u \cdot \nabla v \, dx - \cancel{\int_{\Gamma_N} n \cdot K \nabla u \, v \, dx} = \int_\Omega f v \, dx \qquad \forall v \in V_0 .$$

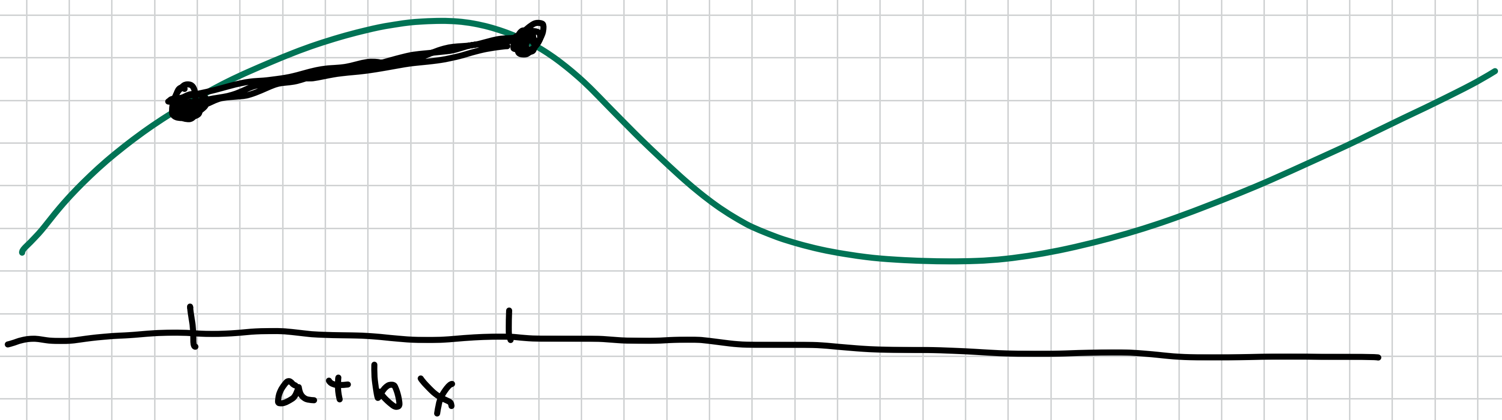
$\to$ Find $u \in V_g$ s.t.

$$\int_\Omega K \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v$$

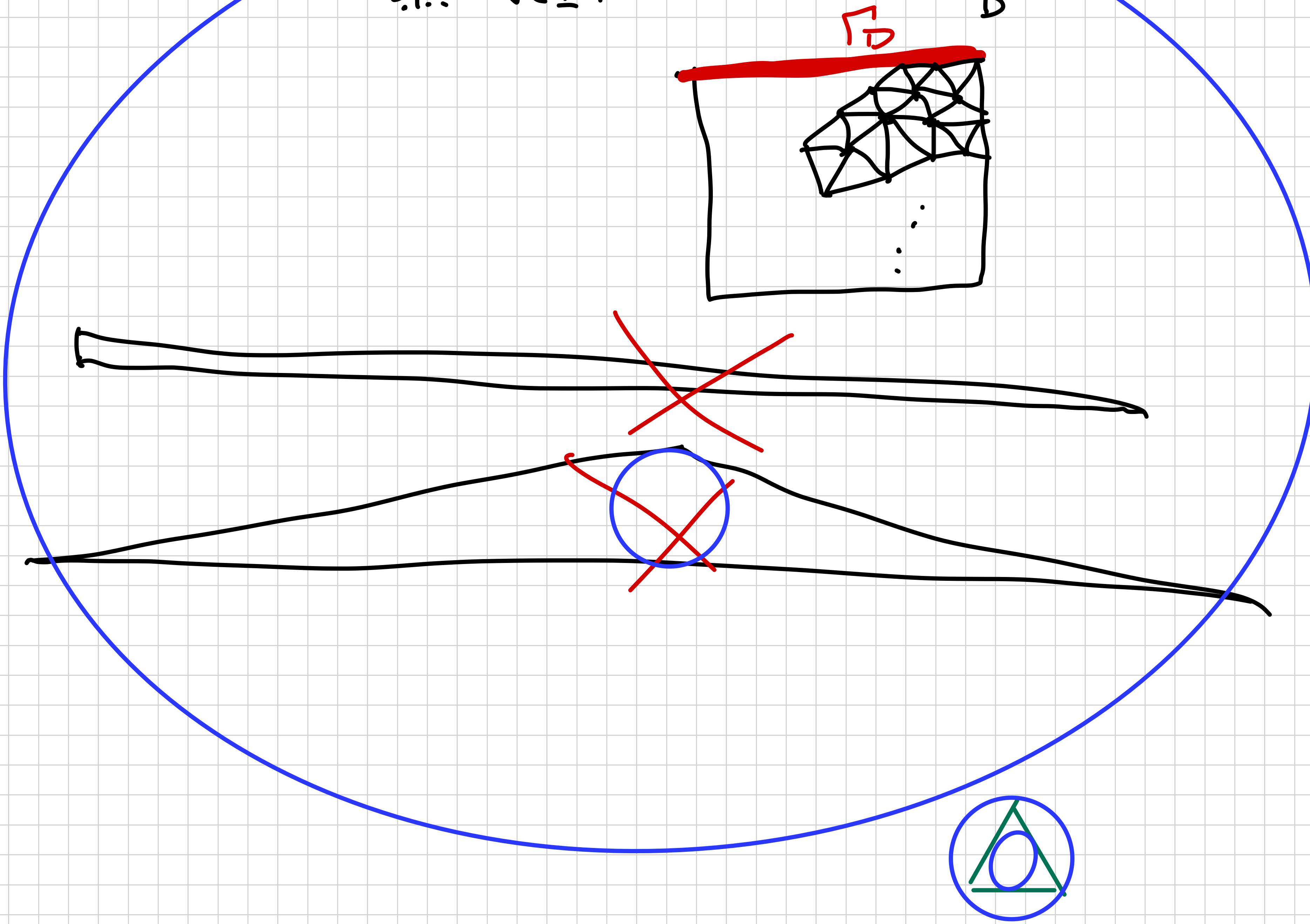
in 1D: V = space of function with $v(0) = 0$.

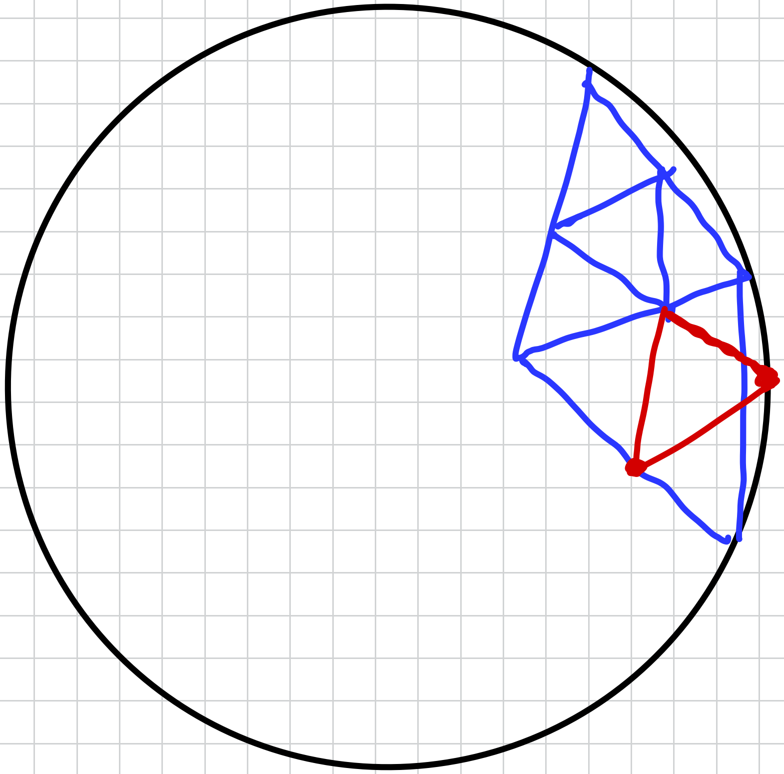
V^h = space of piece-wise linear, continuous functions with $v(0) = 0$.





in 2D: V = space of functions s.t. $v(\underline{x}) = 0$ on Γ_D





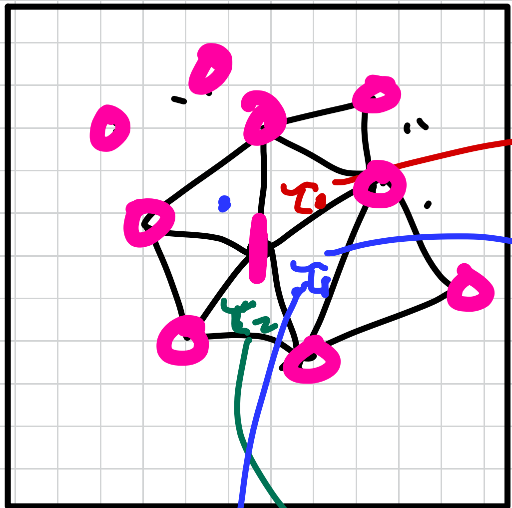
let T^h = triangulation of Ω.

= "nice" triangles
and
"conforming" to $\partial\Omega$.

let V^h = space of linear function (and continuous) on each $\tau \in T^h$.

"tau"

$$= \left\{ v \in C^0(\Omega) \;\middle|\; \begin{array}{l} v = a + bx + cy \\ \text{on each } \tau \in T^h \end{array} \right\}$$

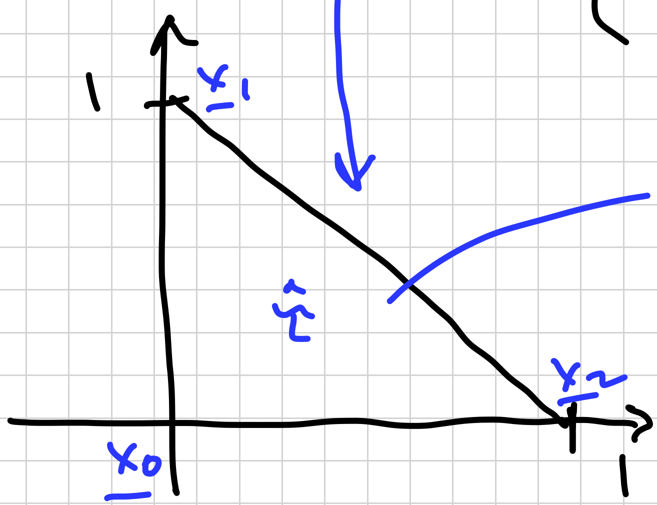


$$u^h(\underline{x})\Big|_{\tau_0} = a_0 + b_0 x + c_0 y$$

$$u^h(x)\Big|_{\tau_1} = a_1 + b_1 x + c_1 y$$

$$u^h(x)\Big|_{\tau_2} = a_2 + b_2 x + c_2 y$$

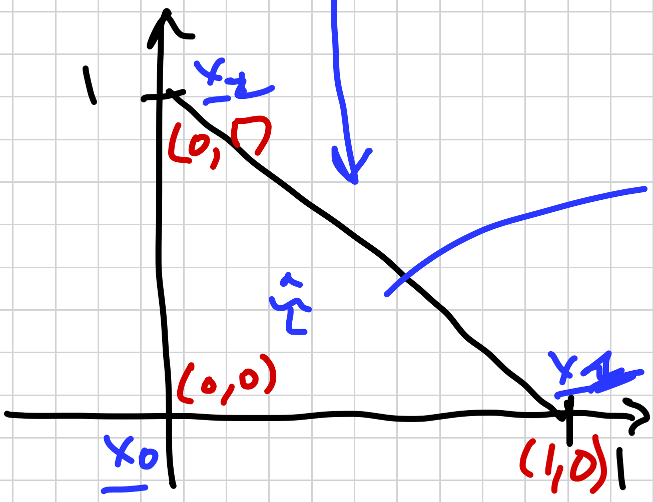
let $\phi_j = \begin{cases} 1 & \text{at } \underline{x} = \underline{x}_j \\ 0 & \text{at all other } \underline{x}_i \end{cases}$



$$\hat{u}(x) = a + bx + cy$$

$$= c_0 \hat{\phi}_0(x) + c_1 \hat{\phi}_1(x) + c_2 \hat{\phi}_2(x)$$

$$\hat{\phi}_j(\underline{x}) = \begin{cases} 1 & \underline{x} = \underline{x}_j \\ 0 & \underline{x} = \underline{x}_i \neq \underline{x}_j \end{cases}$$



$$\hat{u}(x) = a + bx + cy$$
$$= c_0 \hat{\phi}_0(x) + c_1 \hat{\phi}_1(x) + c_2 \hat{\phi}_2(x)$$

$$\hat{\phi}_j(\underline{x}) = \begin{cases} 1 & \underline{x} = \underline{x}_j \\ 0 & \underline{x} = x_i \neq x_j \end{cases}$$

$$\hat{\phi}_0(\underline{x}) = 1 - x - y$$
$$\hat{\phi}_1(\underline{x}) = x$$
$$\hat{\phi}_2(\underline{x}) = y$$

Find $u^h \in V^h$ s.t.

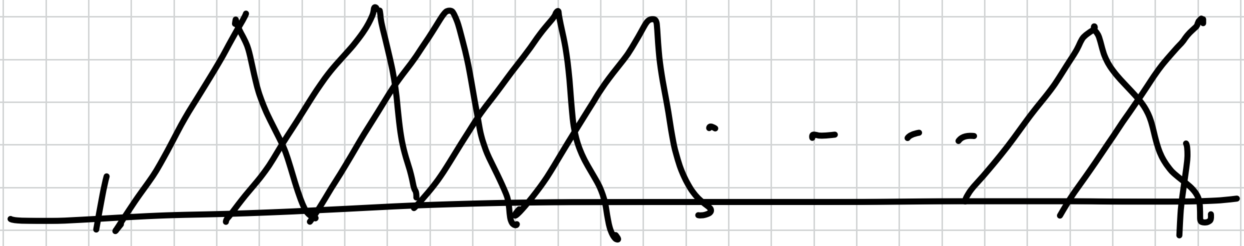
$$\underbrace{\int_\Omega k \nabla u^h \cdot \nabla v^h \, dx}_{a(u^h, v^h)} = \underbrace{\int_\Omega f \, v^h \, dx}_{\langle f, v^h \rangle} \quad \forall v^h \in V^h$$

over elements $\tau \in T^h$:

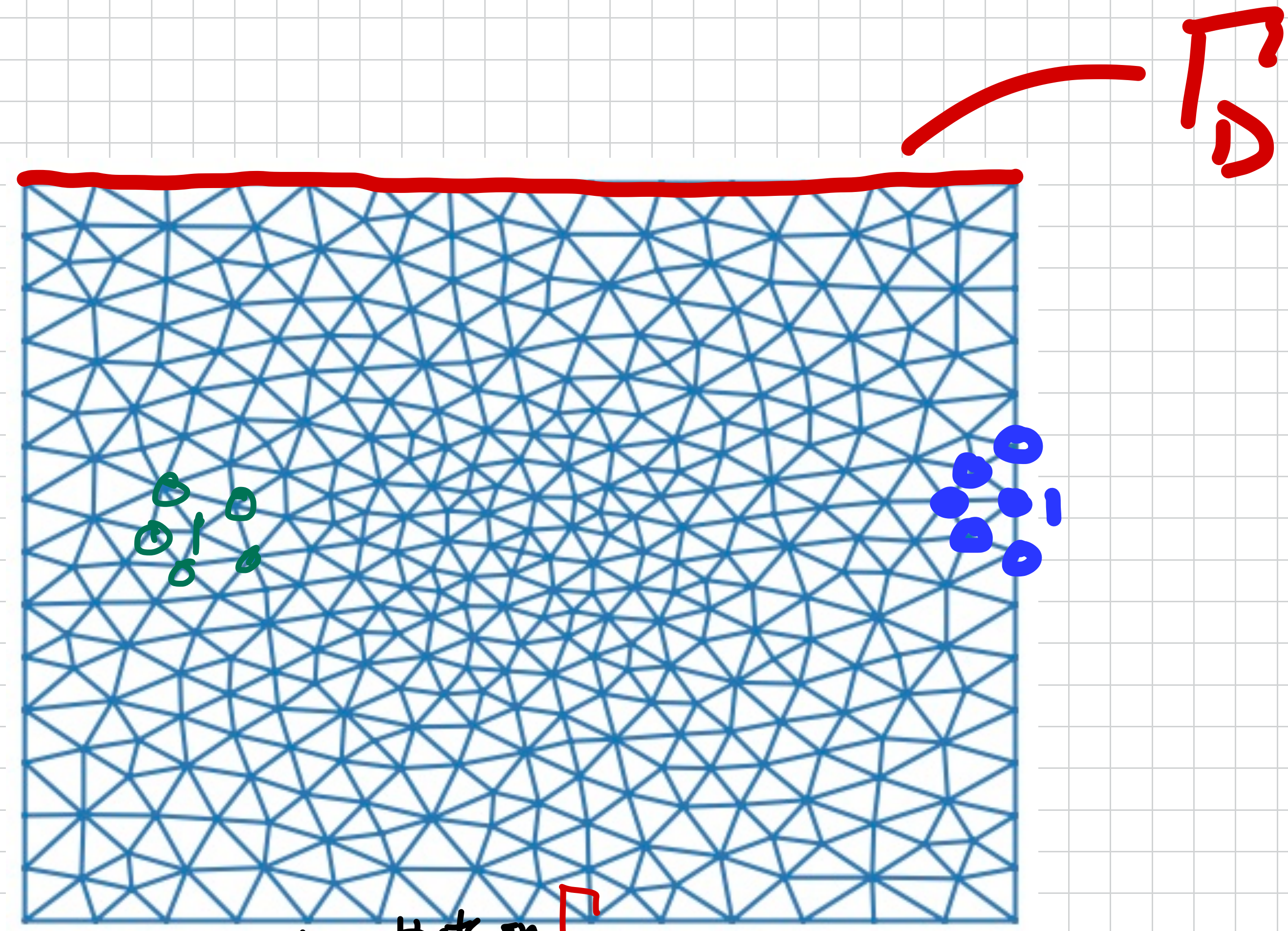
$$\sum_{\tau \in T^h} \int_\tau k \cdot \nabla u^h \cdot \nabla v^h \, dx = \sum_{\tau \in T^h} \int_\tau f \, v^h \, dx$$

Let $\{\phi_j\}$ be a basis for V^h.

1D:



2D:



$$u^h = \sum_{i=0}^{\#\text{pts} - \#\text{pts on } \Gamma_D} c_i \, \phi_i(x)$$

let $A_{ij} = a(\phi_j, \phi_i)$

$$= \sum_{\tau} \int_{\tau} k(x) \nabla \phi_j \cdot \nabla \phi_i \, d\underline{x}$$

$$F_i = \langle f, \phi_i \rangle$$

$$= \sum_{\tau} \int_{\tau} f \, \phi_i(x) \, dx$$

A is "sparse"

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 3 & 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

data $= AA = \begin{bmatrix} 1.0 & 2.0 & 3.0 & 4.0 & 5.0 & 6.0 & 7.0 & 8.0 & 9.0 & 10.0 & 11.0 & 12.0 \end{bmatrix}$

indices $= JA = \begin{bmatrix} 0 & 3 & 0 & 1 & 3 & 0 & 2 & 3 & 4 & 2 & 3 & 4 \end{bmatrix}$

indptr $IA = \begin{bmatrix} 0 & 2 & \cancel{6}\,5 & 9 & 11 & 12 \end{bmatrix}$

index to the $j^{\underline{th}}$ row in "data" and "indices"

nnz entries

column indices, nnz entries