# CS556 Iterative Methods Fall 2024 Homework 3.

Due Tuesday, Oct. 8, 5 PM.

**1. Conjugate Gradient Convergence.** Consider the $n \times n$ SPD matrix

$$A \quad = \quad VDV^T, \qquad (1)$$

where $V$ is orthonormal and $D$=diag(1, 2 … $n-1$, $M$), with $M := n^2$. Let $A_n$ be defined in the same way as $A$, save that $M = n$.

- What is the condition number of $A_n$?
- What is the condition number of $A$?
- Using the CG error bound (e.g., Eq. (25) of the "projection2a.pdf" from the 9-24-2024 notes on Relate), estimate the number of iterations $k$ required to reach a *relative* error tolerance, $tol$=$2 \times 10^{-8}$ in the $A$-norm or $A_n$-norm when solving a system in $A$ or $A_n$, respectively. (Use actual logarithms, rather than the quick estimates we used in class.) Please show the steps used to reach your given estimate.
- **Challenge.** Let $k_{\max}$ and $k_n$ denote your estimated iteration counts from the preceding question. Use the best-fit principles of CG to show that the number of CG iterations for solving $A\underline{x} = \underline{b}$ to tolerance $tol$ is bounded by $k_n + 1$. *Hint: Think carefully about a polynomial that would solve the discrete minimax problem.*
- Code up CG Saad (2nd Ed.) **Alg. 6.18** and solve $A\underline{x} = \underline{b}$ with random rhs $\underline{b}$ with $n$ = 101. What number of iterations do you find to reach a relative $A$-norm error tolerance of $tol$=$2 \times 10^{-8}$? (Note that $V$ can be generated as `[V,R]=qr(rand(n,n));`.
- Make a semilogy plot of $\|\underline{e}_k\|_A / \|\underline{x}\|_A$ and $\|\underline{r}_k\|_2 / \|\underline{b}\|_2$ vs. $k$, both on the same graph, and comment on your observations for the graph and the overall analysis.

**2. Coarse-Grid + Diagonal Preconditioning.**

- Full description to be announced.