(1)

LEC 6

9/12/24

- Jacobi Iteration
- Initially
  ~~k $\underline{x}_0 = \underline{0}$~~

Analysis

$\underline{x}_0 = 0$

for $k = 1:k_{max}$

$x_k = x_{k-1} + D^{-1}(b - Ax_{k-1})$

In practice

$\underline{x} = 0$

for

$\underline{x} = \underline{x} + D^{-1}(\underline{b} - A\underline{x})$

- Two main question re complexity:
  (1) How much work per iteration?
  (2) How many iterations?

- A realistic loop:

$\underline{x} = 0 \quad \underline{r} = \underline{b}$

for $k = 1:k_{max}$

- $\alpha = \|\underline{r}\|_2 = \sqrt{r^T r}$ — $2n$ + comm
- If $\alpha \le$ tol break
- $\underline{x} = \underline{x} + D^{-1}\underline{r}$ — $2n$ $(x_i = x_i + d_i \ast r_i)$

$\underline{r} = \underline{b} - \underbrace{A \cdot \underline{x}}_{2(2d+1)n}$

end

$\underline{x} = 0 \quad \underline{r} = \underline{b}$

for $k = 1:k_{max}$

$\underline{s} = D^{-1}\underline{r}$ — n

$\alpha = \|\underline{s}\|_2$ — $2n$

if $\alpha <$ tol break

$\underline{x} = \underline{x} + \underline{s}$ — n

$\underline{r} = \cancel{b} - A\cancel{x} = b - A(x_{k-1} + s)$ — $2(2d+1)n$

end

$= b - Ax_{k-1} - As = r - As$

n

$2n$

n

$2(2d+1)n$

$d=1$	$10\,n \cdot k$
2	$14\,n \cdot k$
3	$18\,n \cdot k$

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$$-\left[\underline{x}_{k+1} = \underline{x}_k + D^{-1}(\underline{b} - A\underline{x}_k)\right]$$
$$+\left[\underline{x} = \underline{x} + D^{-1}(\underline{b} - A\underline{x})\right]$$

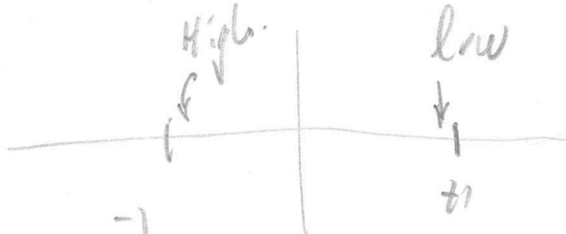
$$\underline{e}_{k+1} = \underline{e}_k - D^{-1}A\,\underline{e}_k$$

$$= (I - D^{-1}A)\,\underline{e}_k$$

$$= G\,\underline{e}_k$$

- Spectral radius of G –
  - Note – high wavenumber of $D^{-1}A = \delta^{-1}_k$ is moved towards origin because of "$-A$"!

1) $D = \frac{2\delta}{h^2} I$



(3)

Start with smallest eigenvalue of A_d

$$\lambda_{min}^{(A_d)} = d\pi^2 + O(h^2)$$

$$\lambda_{max}(G) = \lambda(I - D^{-1}A) = 1 - \frac{h^2}{2d}\cdot\left(d\pi^2 + O(h^2)\right)$$

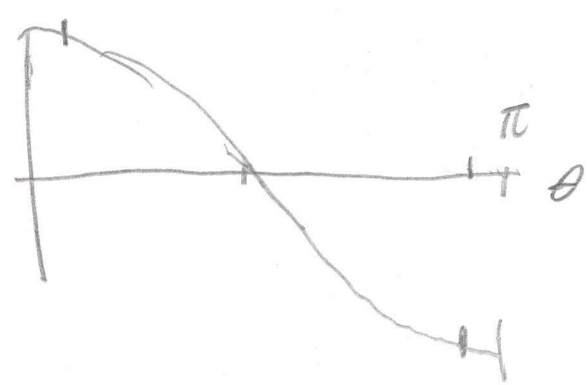
$$\sim 1 - \frac{(\pi h)^2}{2}$$

$$\lambda_{min}(G) = 1 - \frac{h^2}{2d}\lambda_{max}(A)$$

$$\lambda_{max}(A) = d\cdot\frac{2}{h^2}\left(1 - \cos\theta_n\right)$$

$$\theta_k = \frac{\pi k h}{L} = \frac{\pi k}{N} = \frac{\pi k}{n+1}$$

$$\theta_n = \frac{\pi n}{n+1}$$



$$\cos\theta_n = ?$$

$$= -1\cdot\cos\theta_1$$

$$\cos\theta_1 = \cos\frac{\pi h}{L} = 1 - \frac{1}{2}\frac{\pi^2 h^2}{L^2} + \frac{1}{4!}\frac{\pi^4 h^4}{L^4} - \dots$$

$$\therefore \lambda_{max}(A) = \frac{d2}{h^2}\left(1 - \cos\theta_N\right)$$

$$= \frac{2d}{h^2}\left[2 - \frac{\pi^2 h^2}{2} + O(h^4)\right]$$

$$\lambda_{min}(G) = 1 - \frac{h^2}{2d}\cdot\left[\frac{2d}{h^2}\left(2 - \frac{\pi^2 h^2}{2} + O(h^4)\right)\right]$$

$$= -1 + \frac{\pi^2 h^2}{2} + O(h^4)$$

$$|\lambda_{min}(G)| = 1 - \frac{\pi^2 h^2}{2} + O(h^4) \sim |\lambda_{max}(G)|$$

$$\rho(G) \sim 1 - \frac{\pi^2 h^2}{2} \quad \left(\sim 1 \ ☹\right)$$

Vector Norms

Consider a vector norm: $\|\underline{x}\|_*$ - $(\underline{x} \in \mathbb{R}^n)$

Ex: i $\|\underline{x}\|_1 = \sum_{j=1}^{n} |x_j|$

ii $\|\underline{x}\|_\infty = \max_j |x_j|$

iii $\|\underline{x}\|_p := \left[\sum_{j=1}^{n} |x_j|^p \right]^{\frac{1}{p}}$

iv $\|\underline{x}\|_2 = \left[\sum_{j=1}^{n} x_j^2 \right]^{\frac{1}{2}} = \sqrt{(\underline{x}^T \underline{x})}^{\frac{1}{2}}$

If A SPD, can also ~~consider~~ define

(v) $\|\underline{x}\|_A = (\underline{x}^T A \underline{x})^{\frac{1}{2}}$

For (iv) and (v), we also have an associated inner product

$(\underline{u}, \underline{v}) := \underline{u}^T \underline{v} \longrightarrow \|\underline{u}\|_2 = (\underline{u}, \underline{u})^{\frac{1}{2}}$

$(\underline{u}, \underline{v})_A := \underline{u}^T A \underline{v} \qquad \|\underline{u}\|_A = (\underline{u}^T, \underline{u})_A^{\frac{1}{2}}$

⑥

equivalence of norms.

- For n fixed, every pair of vector norms satisfies the following equivalence ~~of norms~~ property:

  There exist constants c and C, dependent on n, $\|\cdot\|_*$, $\|\cdot\|_0$ but not on $\underline{x}$ s.t.

$$c\,\|\underline{x}\|_0 \le \|\underline{x}\|_* \le C\,\|\underline{x}\|_0$$

- What this says is that, if $\|\underline{x}_k\|_0 \to 0$ then $\|x_k\|_* \to 0$.

- That is, if we bound the error in one norm as $k \to \infty$, then it is bounded in all other vector norms (n-fixed!).

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Matrix Norms

- We will work almost exclusively with induced matrix norms, defined as follows

$$\|A\|_\alpha = \max_{\underline{x}\in\mathbb{R}^n} \frac{\|A\underline{x}\|_\alpha}{\|\underline{x}\|_\alpha} = \max_{\substack{\underline{x}\in\mathbb{R}^n \\ \|x\|_\alpha = 1}} \|A\underline{x}\|_\alpha$$

- $\|A\|_\alpha \geq 0$ ~~measures~~ indicates the maximum possible "stretching" of any input vector $\underline{x}$.

~~$\|A\|_\alpha$~~

- Of course we can ~~not~~ have $\|A\|_\alpha < 1$,
- Moreover $\|A\|_\alpha = 0$ only for $A = 0$.
- If $A = A^T$, $\|A\|_2 = \rho(A)$
- $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$ (max row sum)
- $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$ (max col sum)

- Let's look at the utility of $\|A\|_*$.
- Return to our convergence rate question:

$$\underline{e}_{k+1} = G\,\underline{e}_k \quad , \quad \underline{e}_0 = \underline{x} - \underline{x}_0 = \underline{x}$$

$$\|e_{k+1}\|_* = \|G\,\underline{e}_k\|_*$$

$$\leq \|G\|_* \|e_k\|_* \qquad (\text{Def'n of } \|G\|_*)$$

$$\leq \|G\|_*^{k+1} \|e_0\|_*$$

$$= \|G\|_*^{k+1} \|\underline{x}\|$$

$\therefore$ Relative error is bounded by

$$\boxed{\frac{\|e_k\|_*}{\|\underline{x}\|} \leq \|G\|_*^k}$$

⑨

Note - for uniform grid case, $D(A) = a_{11} \cdot I$

$$\therefore \quad G := I - D^{-1}A = I - \frac{1}{a_{11}}A$$

$$= I^T - \frac{1}{a_{11}}A^T = G^T$$

$$\therefore \quad \|G\|_2 = \rho(G)$$

So, for any $\underline{x}$ and initial guess $\underline{x}_0 = \underline{0}$,

$$\frac{\|\underline{e}_k\|_2}{\|\underline{x}\|_2} \le \rho(G)^k$$

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For our particular case

$$\rho(G) = 1-\epsilon \qquad = 1 - \frac{\pi^2 h^2}{2} + O(h^4)$$

Q: What k for

$$(1-\epsilon)^k \le tol := 10^{-6} \;?$$

Note that $e^{-\epsilon} = 1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{3!} \cdots$

$$(1-\epsilon)^k \approx (e^{-\epsilon})^k = e^{-k\epsilon} \simeq 10^{-6} \simeq e^{-12}$$

$$\boxed{k \simeq \frac{12}{\epsilon} = 12\left(\frac{2}{\pi^2 h^2}\right) = \frac{24}{10} \cdot N^{-2} \simeq 2.5\, N^2}$$

(11)

- Amount of work / iter:

$$w = (10 + 4d) \cdot n = (10 + 4d) N^d \quad \begin{cases} 9 \\ 13 \\ 17 \end{cases}$$

- # Iters: $k \sim 2.5 N^2$

$$W = wk = \begin{cases} 35 \; N^3 \\ N^4 \\ N^5 \end{cases}$$