CS556 Iterative Methods Fall 2024 Quiz 4.

Due Tuesday, Sep. 17, 5 PM.

1. Matrix Norms

Suppose you wish to solve $H\underline{u} = \underline{f}$ with

$$H \;=\; I \;+\; \nu\Delta t A, \tag{1}$$

where A is the centered finite-difference approximation to the 1D Poisson problem with grid spacing h and $\nu\Delta t = h^2/2$.

- Use a matrix norm to estimate the number of Jacobi iterations to reduce the relative error by 10^{-6} as a function of n. Show your work.
- For a fixed n, write a small test code and plot the error as a function of iteration count, k, along with the error bound from your estimate above.

2. Jacobi vs. Block-Jacobi

The following is an *analysis* question—all work should be done by hand. (You can of course compute to confirm your estimates, if you so desire.)

Consider a centered finite-difference approximation to $-\nabla^2 u = f$ on $\Omega = [0,1]^2$ with homogeneous Dirichlet conditions using uniform grid-spacing, h, in each direction, with $h = L/(m+1)$ (where $L = 1$ in this case), and $n = m^2$. If a lexicographical ordering of unknowns is used, the resultant system, $A\underline{u} = \underline{f}$, is characterized by the matrix,

$$A \;=\; I_y \otimes A_x \;+\; A_y \otimes I_x, \tag{2}$$

where $I_y = I_x$ is the $m \times m$ identity matrix and where $A_y = A_x$ is the $m \times m$ tridiagonal matrix, $A_x = \frac{1}{h^2}$tridiag(-1,2,-1).

2a. What is the spectral radius of the error propagator, $E = I - D^{-1}A$, for the standard *pointwise* Jacobi preconditioner, D =diag(a_{ii}) as a function of n? ($\rho(E) \sim 1 + Cn^k$, what are C and k?)

We know that A has a block-tridiagonal form,

$$A \;=\; \frac{1}{h^2}\text{block-tridiag}(-I_x, H_x, -I_x), \tag{3}$$

where the inner block, $H_x := 2I_x + h^2 A_x$ is itself tridiagonal. It is proposed to use **block Jacobi** iteration (e.g., Saad, Sec. 4.1.1) of the form

$$\underline{u}_k \;=\; \underline{u}_{k-1} \;+\; H^{-1}(\underline{b} \;-\; A\underline{u}_{k-1}), \tag{4}$$

with $H = \frac{1}{h^2}\left(I_y \otimes H_x\right)$. (**Note:** This is not the *only* possible block-Jacobi choice.)

2b. What is the amount of work per iteration in this block-Jacobi case? (Justify your estimate–don't just provide a number.)

2c. What is the spectral radius of the error propagator, $E = I - H^{-1}A$, in this case? (Answer in the same form as for **2a.**)

2d. How much savings would you expect for the case of $n = 10000$ (i.e., $m = 100$) from this strategy?

Hint: H and A share the same eigenvectors.