CS 598 EVS: Tensor Computations

Matrix Computations Background

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Conditioning

- **Absolute Condition Number**:

- **(Relative) Condition Number**:

Posedness and Conditioning

- **What is the condition number of an ill-posed problem?**

Matrix Condition Number

- The matrix condition number $\kappa(\boldsymbol{A})$ is the ratio between the max and min distance from the surface to the center of the unit ball (norm-1 vectors) transformed by $\boldsymbol{A}$:

- The matrix condition number bounds the worst-case amplification of error in a matrix-vector product:

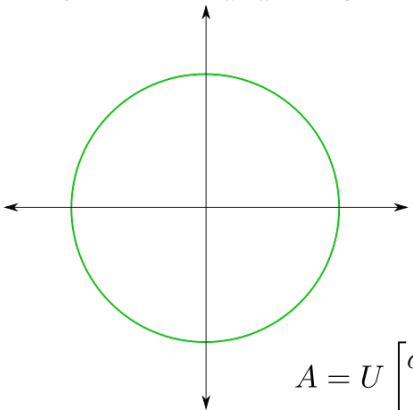
Singular Value Decomposition

- The singular value decomposition (SVD)

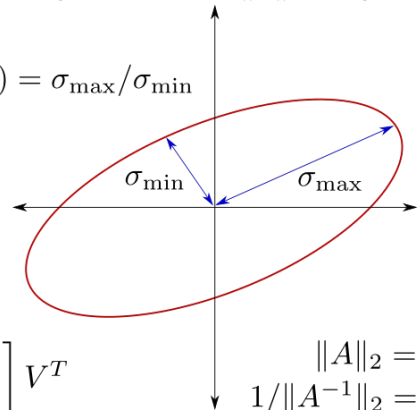
- Condition number in terms of singular values

Visualization of Matrix Conditioning

$$\{x : x \in \mathbb{R}^2, \|x\|_2 = 1\} \xrightarrow{A} \{Ax : x \in \mathbb{R}^2, \|x\|_2 = 1\}$$



$$\kappa(A) = \sigma_{\max}/\sigma_{\min}$$



$$A = U \begin{bmatrix} \sigma_{\max} & \\ & \sigma_{\min} \end{bmatrix} V^T$$

$$\|A\|_2 = \sigma_{\max}$$
$$1/\|A^{-1}\|_2 = \sigma_{\min}$$

Linear Least Squares

- Find $\boldsymbol{x}^\star = \text{argmin}_{\boldsymbol{x}\in\mathbb{R}^n} ||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_2$ where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$:

- Given the SVD $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ we have $\boldsymbol{x}^\star = \underbrace{\boldsymbol{V}\boldsymbol{\Sigma}^\dagger\boldsymbol{U}^T}_{\boldsymbol{A}^\dagger}\boldsymbol{b}$, where $\boldsymbol{\Sigma}^\dagger$ contains the reciprocal of all nonzeros in $\boldsymbol{\Sigma}$, and more generally $\dagger$ denotes pseudoinverse:

Normal Equations

Demo: *Normal equations vs Pseudoinverse*
Demo: *Issues with the normal equations*

- *Normal equations* are given by solving $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^T\boldsymbol{b}$:

- However, solving the normal equations is a more ill-conditioned problem then the original least squares algorithm

Solving the Normal Equations

- If $\boldsymbol{A}$ is full-rank, then $\boldsymbol{A}^T\boldsymbol{A}$ is symmetric positive definite (SPD):

- Since $\boldsymbol{A}^T\boldsymbol{A}$ is SPD we can use Cholesky factorization, to factorize it and solve linear systems:

QR Factorization

- If $\boldsymbol{A}$ is full-rank there exists an orthogonal matrix $\boldsymbol{Q}$ and a unique upper-triangular matrix $\boldsymbol{R}$ with a positive diagonal such that $\boldsymbol{A} = \boldsymbol{QR}$

- A reduced QR factorization (unique part of general QR) is defined so that $\boldsymbol{Q} \in \mathbb{R}^{m \times n}$ has orthonormal columns and $\boldsymbol{R}$ is square and upper-triangular

- We can solve the normal equations (and consequently the linear least squares problem) via reduced QR as follows

Computing the QR Factorization

- The Cholesky-QR algorithm uses the normal equations to obtain the QR factorization

- Orthogonalization-based methods are most efficient and stable for QR factorization of dense matrices

Eigenvalue Decomposition

- If a matrix $\boldsymbol{A}$ is diagonalizable, it has an *eigenvalue decomposition*

- $\boldsymbol{A}$ and $\boldsymbol{B}$ are *similar*, if there exist $\boldsymbol{Z}$ such that $\boldsymbol{A} = \boldsymbol{Z}\boldsymbol{B}\boldsymbol{Z}^{-1}$

Similarity of Matrices

matrix	*similarity*	*reduced form*
SPD		
real symmetric		
Hermitian		
normal		
real		
diagonalizable		
arbitrary		

Rayleigh Quotient

- For any vector x that is close to an eigenvector, the *Rayleigh quotient* provides an estimate of the associated eigenvalue of A:

Introduction to Krylov Subspace Methods

- *Krylov subspace methods* work with information contained in the $n \times k$ matrix

$$\boldsymbol{K}_k = \begin{bmatrix} \boldsymbol{x_0} & \boldsymbol{A x_0} & \cdots & \boldsymbol{A}^{k-1}\boldsymbol{x_0} \end{bmatrix}$$

- $\boldsymbol{A}$ is similar to *companion matrix* $\boldsymbol{C} = \boldsymbol{K}_n^{-1}\boldsymbol{A}\boldsymbol{K}_n$:

Krylov Subspaces

- Given $\boldsymbol{Q}_k\boldsymbol{R}_k = \boldsymbol{K}_k$, we obtain an orthonormal basis for the Krylov subspace,

$$\mathcal{K}_k(\boldsymbol{A}, \boldsymbol{x}_0) = span(\boldsymbol{Q}_k) = \{p(\boldsymbol{A})\boldsymbol{x}_0 : deg(p) < k\},$$

  where p is any polynomial of degree less than k.
- The Krylov subspace includes the $k-1$ approximate dominant eigenvectors generated by $k-1$ steps of power iteration:

Krylov Subspace Methods

- The $k \times k$ matrix $\boldsymbol{H}_k = \boldsymbol{Q}_k^T \boldsymbol{A} \boldsymbol{Q}_k$ minimizes $||\boldsymbol{A}\boldsymbol{Q}_k - \boldsymbol{Q}_k \boldsymbol{H}_k||_2$:

- $\boldsymbol{H}_k$ is upper-Hessenberg, because the companion matrix $\boldsymbol{C}_n$ is upper-Hessenberg:

Rayleigh-Ritz Procedure

- The eigenvalues/eigenvectors of $\boldsymbol{H}_k$ are the *Ritz values/vectors*:

- The Ritz vectors and values are the *ideal approximations* of the actual eigenvalues and eigenvectors based on only $\boldsymbol{H}_k$ and $\boldsymbol{Q}_k$:

Low Rank Matrix Approximation

- Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ seek rank $r < m, n$ approximation

- Eckart-Young (optimal low-rank approximation by SVD) theorem

Rank Revealing Matrix Factorizations

- Computing the SVD

- QR with column pivoting

Orthogonal Iteration

- For sparse matrices, QR factorization creates fill, so must revert to iterative methods

- Orthogonal iteration interleaves deflation and power iteration

Randomized SVD

- Orthogonal iteration for SVD can also be viewed as a randomized algorithm

Generalized Nyström Algorithm

- The generalized Nyström algorithm provides an efficient way of computing a sketched low-rank factorization

Multidimensional Optimization

- Minimize $f(\boldsymbol{x})$

- Quadratic optimization $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}$

Basic Multidimensional Optimization Methods

- Steepest descent: minimize f in the direction of the negative gradient:

- Given quadratic optimization problem $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x}$ where $\boldsymbol{A}$ is symmetric positive definite, the error $\boldsymbol{e}_k = \boldsymbol{x}_k - \boldsymbol{x}^*$ satisfies

  $$||\boldsymbol{e}_{k+1}||_{\boldsymbol{A}} =$$

  - When sufficiently close to a local minima, general nonlinear optimization problems are described by such an SPD quadratic problem.
  - Convergence rate depends on the conditioning of $\boldsymbol{A}$, since

Gradient Methods with Extrapolation

- We can improve the constant in the linear rate of convergence of steepest descent by leveraging *extrapolation methods*, which consider two previous iterates (maintain *momentum* in the direction $\boldsymbol{x}_k - \boldsymbol{x}_{k-1}$):

- The *heavy ball method*, which uses constant $\alpha_k = \alpha$ and $\beta_k = \beta$, achieves better convergence than steepest descent:

Conjugate Gradient Method

- The *conjugate gradient method* is capable of making the optimal (for a quadratic objective) choice of α_k and β_k at each iteration of an extrapolation method:

- *Parallel tangents* implementation of the method proceeds as follows

Krylov Optimization

- Conjugate gradient (CG) finds the minimizer of $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}$ (which satisfies optimality condition $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$) within the Krylov subspace of $\boldsymbol{A}$:

CG and Krylov Optimization

The solution at the kth step, $\boldsymbol{y}_k = ||\boldsymbol{b}||_2 \boldsymbol{T}_k^{-1} \boldsymbol{e}_1$ is obtained by CG from $\boldsymbol{y}_{k+1}$ with a single matrix-vector product with $\boldsymbol{A}$ and vector operations with $O(n)$ cost

Preconditioning

- Convergence of iterative methods for $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ depends on $\kappa(\boldsymbol{A})$, the goal of a preconditioner $\boldsymbol{M}$ is to obtain $\boldsymbol{x}$ by solving

$$\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{M}^{-1}\boldsymbol{b}$$

  with $\kappa(\boldsymbol{M}^{-1}\boldsymbol{A}) < \kappa(\boldsymbol{A})$

- Common preconditioners select parts of $\boldsymbol{A}$ or perform inexact factorization

Conjugate Gradient Convergence Analysis

- In previous discussion, we assumed $\boldsymbol{K}_n$ is invertible, which may not be the case if $\boldsymbol{A}$ has $m < n$ distinct eigenvalues, however, in exact arithmetic CG converges in $m - 1$ iterations[1]

[1] This derivation follows *Applied Numerical Linear Algebra* by James Demmel, Section 6.6.4

Conjugate Gradient Convergence Analysis (II)

- Using $\boldsymbol{z} = \boldsymbol{\rho}_{k-1}(\boldsymbol{A})\boldsymbol{A}\boldsymbol{x}$, we can simplify $\phi(\boldsymbol{z}) = (\boldsymbol{x} - \boldsymbol{z})^T \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{z})$ as

- We can bound the objective based on the eigenvalues of $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ using the identity $\boldsymbol{p}(\boldsymbol{A}) = \boldsymbol{Q}\boldsymbol{p}(\boldsymbol{\Lambda})\boldsymbol{Q}^T$,

Conjugate Gradient Convergence Analysis (III)

- Using our bound on the square of the residual norm $\phi(\boldsymbol{z})$, we can see why CG converges after $m-1$ iterations if there are only $m < n$ distinct eigenvalues

- To see that the residual goes to 0, we find a suitable polynomial in $\mathcal{Q}_m$ (the set of polynomials q_m of degree m with $q_m(0) = 1$)

Round-off Error in Conjugate Gradient

- CG provides strong convergence guarantees for SPD matrices in exact arithmetic

- Due to round-off CG may stagnate / have plateaus in convergence

Graph and Matrix Duality

- graphs have have a natural correspondence with sparse matrices

- matrix-based representations of graphs can be used to devise algorithms

Graph Partitioning from Eigenvectors

- The Laplacian matrix provides a model of interactions on a graph that is useful in many contexts

- The second-smallest-eigenvalue eigenvector of the Laplacian (the Fiedler vector), gives a good partitioning of the graph

Newton's Method

- Newton's method in n dimensions is given by finding minima of n-dimensional quadratic approximation using the gradient and Hessian of f:

Nonlinear Least Squares

- An important special case of multidimensional optimization is *nonlinear least squares*, the problem of fitting a nonlinear function $f_{\boldsymbol{x}}(t)$ so that $f_{\boldsymbol{x}}(t_i) \approx y_i$:

- We can cast nonlinear least squares as an optimization problem to minimize residual error and solve it by Newton's method:

Gauss-Newton Method

- The Hessian for nonlinear least squares problems has the form:

- The *Gauss-Newton* method is Newton iteration with an approximate Hessian:

Constrained Optimization Problems

- We now return to the general case of *constrained* optimization problems:

$$\min_{\boldsymbol{x}} f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{g}(\boldsymbol{x}) = \mathbf{0} \quad \text{and} \quad \boldsymbol{h}(\boldsymbol{x}) \leqslant \mathbf{0}$$

equality inequality

$Ax = b$ $x \geq 0$

- Generally, we will seek to reduce constrained optimization problems to a series of simpler optimization problems:

Lagrangian Duality

- The Lagrangian function with constraints $\boldsymbol{g}(\boldsymbol{x}) = \mathbf{0}$ and $\boldsymbol{h}(\boldsymbol{x}) \leqslant \mathbf{0}$ is

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^T \begin{bmatrix} h(x) \\ g(x) \end{bmatrix}$$

- The Lagrangian dual problem is an unconstrained optimization problem:

$$\max_{\boldsymbol{\lambda}} q(\boldsymbol{\lambda}), \quad q(\boldsymbol{\lambda}) = \begin{cases} \min_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) & \text{if } \boldsymbol{\lambda} \geqslant \mathbf{0} \\ -\infty & \text{otherwise} \end{cases}$$

strong duality

q^*

$f^* = q^*$

The unconstrained optimality condition $\nabla q(\boldsymbol{\lambda}^*) = \mathbf{0}$, implies

$$\max\left(\lambda^*, \begin{bmatrix} h(x^*) \\ g(x^*) \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathcal{L}(x^*, \lambda^*) = f^* = q^*$$

Optimality and Complementarity Slackness Condition

Consider the inequality-constrained optimization problem, $\boldsymbol{h}(\boldsymbol{x}) \leqslant \mathbf{0}$,

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{h}(\boldsymbol{x})$$

- The pair $\boldsymbol{x}^*$ and $\boldsymbol{\lambda}^*$ are a primal-dual optimal solution $\boldsymbol{x}^*$ is feasible, $\boldsymbol{\lambda}^* \geqslant 0$, and strong duality holds, $f(\boldsymbol{x}^*) = q(\boldsymbol{\lambda}^*) = \mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$

$$\mathcal{L}(x^*, \lambda^*) = f(x^*) - \underbrace{\lambda^{*T} h(x^*)}_{0}$$

$\lambda_i = 0$ or $h_i(x^*) = 0$

Sequential Quadratic Programming

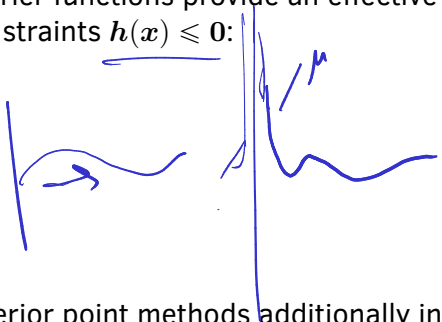
- *Sequential quadratic programming (SQP)* reduces a nonlinear equality constrained problem to a sequence of constrained quadratic programs via a Taylor expansion of the Lagrangian function $\mathcal{L}_f(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{g}(\boldsymbol{x})$:

$$H_{\mathcal{L}_f} = \begin{matrix} x \\ \lambda \end{matrix} \begin{bmatrix} H_f(x) & J_g^T(x) \\ J_g(x) & 0 \end{bmatrix}$$

- SQP ignores the constant term $\mathcal{L}_f(\boldsymbol{x}_k, \boldsymbol{\lambda}_k)$ and minimizes $\boldsymbol{s}$ while treating $\boldsymbol{\delta}$ as a Lagrange multiplier:

Interior Point Methods

- Barrier functions provide an effective way of working with inequality constraints $\boldsymbol{h}(\boldsymbol{x}) \leqslant \mathbf{0}$:



- Interior point methods additionally incorporate Lagrangian optimization

Karush-Kuhn-Tucker (KKT) conditions

Consider the linear-constrained Quadratic program (QP): Its Lagrangian

$f(x) = \frac{1}{2} x^T H x + x^T b$ $\quad Ax = b \quad Cx \geq d$ | $\mathcal{L}(x, \lambda, v) = \frac{1}{2} x^T H x + x^T c - \lambda(Ax - b) - v(Cx - d)$

function may be used to derive an interior point method The first-order optimality (KKT) conditions are

$\nabla \mathcal{L}_x = 0$

$\lambda, v \geq 0$

$Ax - b = 0$

$Cx - d \geq 0 \Rightarrow Cx - d - s = 0$

$s \geq 0 \qquad v^T(Cx - d) = 0$

Primal-dual Interior Point Method (IPM)

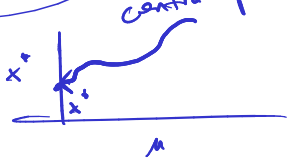
Solve perturbed KKT conditions after introducing slack variables $s \in \mathbb{R}^{m_2}$

μ - barrier parameter

$$\nabla_x f = 0 \quad | \quad s, \lambda, v \geq 0 \quad | \quad Ax - b = 0 \quad | \quad Cx - d - s = 0$$

$$s_i \cdot v_i = \mu \qquad \langle s, v \rangle = \frac{\langle s, v \rangle}{\dim(s)}$$

as $\mu \to 0$ x^* for KKT goes to x^* for original problem



Interior Point Method (IPM): KKT system

Newton's method applied to KKT equations results in linear systems

$$J_{\nabla f(x_k)}\, s_k = -\nabla f(x_k)$$

H_f

Newton for minimizing $f(x)$ = Newton for solving NLS with $\nabla f(x) = 0$

These linear systems become ill-conditioned as the interior point method approaches converges

$$F = \begin{bmatrix} -H & A^T & C \\ A & & \\ C & & D^{(k)} \end{bmatrix} \begin{bmatrix} dx \\ d\lambda \\ du \end{bmatrix}$$

$x \quad \lambda \quad u$

SU^{-1}

$$D^{(k)} = \begin{bmatrix} s_1/u_1 & & \\ & \ddots & \\ & & s_n/u_n \end{bmatrix}$$

$f(x) = \frac{1}{2} x^T H x$

$A \in \mathbb{R}^{m \times n}$

$H \in \mathbb{R}^{n \times n}$ $\quad C \in \mathbb{R}^{n \times n}$

with $C = I$

$Ax = b$

$x \geq 0 \Rightarrow Cx \geq 0$