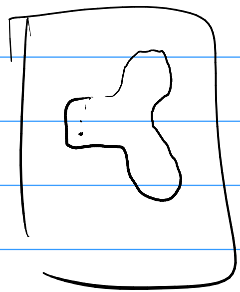
Today

- Taylor: local, multipole, low rank
- linear algebra



3 Rank and Smoothness

Punchline

What do (numerical) rank and smoothness have to do with each other?

Recap: Multivariate Taylor

How does Taylor's theorem get generalized to multiple dimensions?

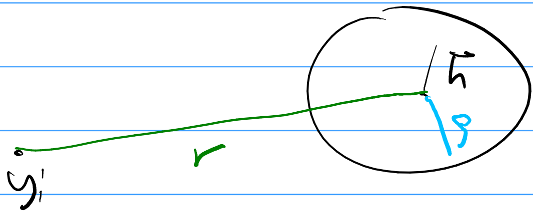
$$f(\vec{x}) = G(\vec{x}_1, \vec{y}_i)$$

$$G(\vec{c} + \vec{h}_1, \vec{y}_i) = f(\vec{c} + \vec{h}) \approx \sum_{|\nu| \leq k} \frac{D^\nu f(\vec{c})}{\nu!} \cdot \vec{h}^\nu$$

$$D^{(0,0)} G(\vec{x}_1, \vec{y}_i) =$$

$$D^{(1,0)} G(\vec{x}_1, \vec{y}_i) = \partial_{x_1} G(\vec{x}_1, \vec{y}_i)$$

$$|D^\nu_x G(\vec{x}_1, y_i)| \leq C \frac{1}{r^{|\nu|}} (|\nu|)!! \quad r = |x - y|_2$$



Taylor remainder

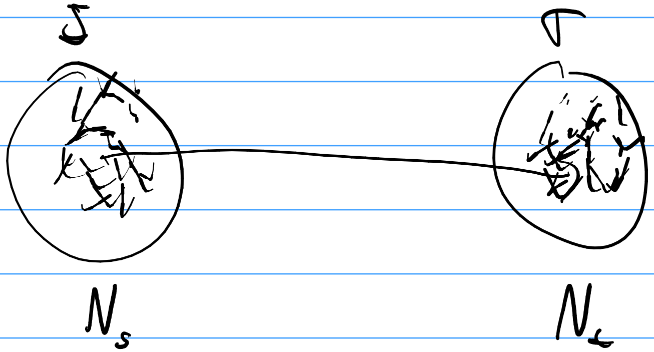
$$\left| f(x) - \underbrace{\sum_{p=0}^{k} \frac{f^{(p)}(c)}{p!} \cdot h^k}_{\le \alpha^p} \right| \le \sum_{p=k+1}^{\infty} \alpha^p \le C\alpha^{k+1}$$

$$\left| G(x,y_i) - \sum_{|\nu|\le k} \underbrace{\overbrace{\frac{D_i^{\nu} G(x,y_i)}{\nu!}}^{\le C \cdot \frac{1}{r^{|\nu|}}} \, h^{\nu}}_{\le \alpha^p} \right|$$

$$\le \sum_{|\nu|>k} \left| \frac{D_i^{\nu} G(x,y_i)}{\nu!} \, h^{\nu} \right|$$

$$\leq C \sum_{|\nu| > k} \left| \frac{1}{r^{|\nu|}} \cdot \rho^{|\nu|} \right|$$

$$\leq C' \left(\frac{\rho}{r} \right)^{k+1}$$



Computational tool:

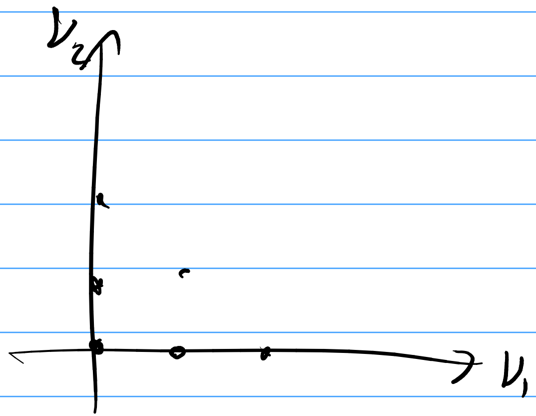
- direct ev. $N_s N_t$
- form Taylor: $k N_s$
- eval Taylor $k N_t$

$$\text{Error} \leq C \cdot \left(\frac{d(c, \text{furthest tgt})}{d(c, \text{closest src})} \right)^{k+1}$$

How many terms in the Taylor series?

$$|\nu_1| + |\nu_2| \leq k$$

Rank: $\sim k^2$



Estimating the rank:

$$\varepsilon = \left(\frac{\text{dist}(c, \text{furthest tgt})}{\text{dist}(c, \text{closest src})}\right)^{k+1} = \rho^{k+1}$$

$$\text{rank} = k^2 \iff \sqrt{\text{rank}} = k$$

$$\varepsilon = \rho^{\sqrt{\text{rank}}+1}$$

$$\text{rank} \approx \left(\frac{\log \varepsilon}{\log \rho} - 1\right)^2$$

Taylor and Error

How can we estimate the error in a Taylor expansion?

Connect Taylor and Low Rank

Can Taylor help us establish low rank of an interaction?

Taylor on Potentials

Compute a Taylor expansion of a 2D Laplace point potential.

Local expansions as a Computational Tool

Low rank makes evaluating interactions cheap(er). Do local expansions help with that goal?

Taylor on Potentials, Again

Stare at that Taylor formula again.