TODAY

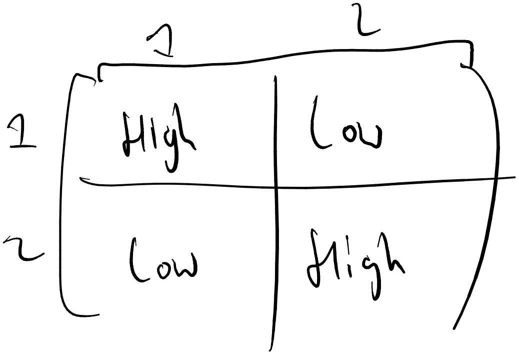
Direct solver

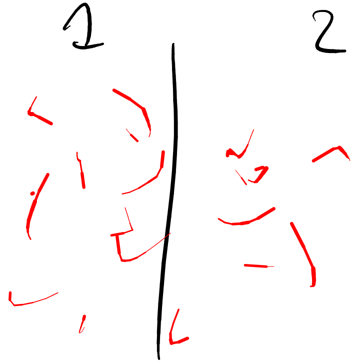
PDE BVPs

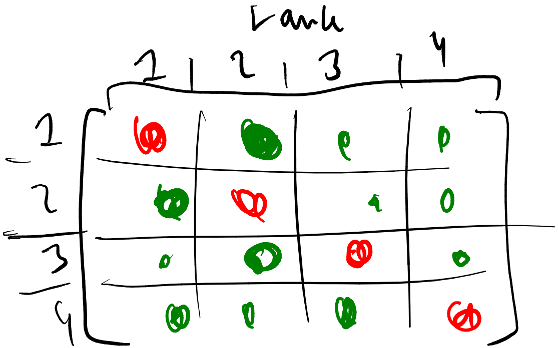
Analysis

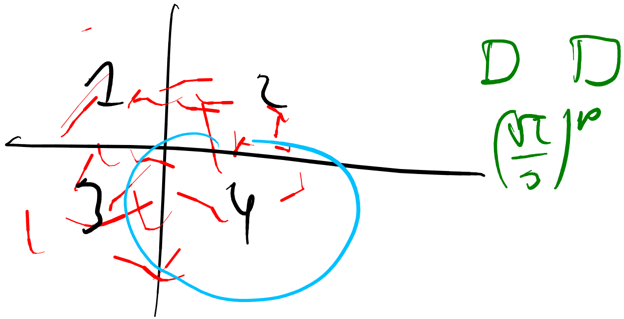
A Matrix View of Low-Rank Interaction

Only *parts of the matrix are low-rank*! What does this look like from a matrix perspective?









Block-separable matrices

How do we represent the low-rank structure of a matrix like this?

$$A = \begin{pmatrix} D_1 & A_{12} & A_{13} & A_{14} \\ A_{21} & D_2 & A_{23} & A_{24} \\ A_{31} & A_{32} & D_3 & A_{34} \\ A_{41} & A_{42} & A_{43} & D_4 \end{pmatrix}$$

where A_{ij} has low-rank structure?

$$A_{compressed} = \begin{pmatrix} D_1 & & & P_1 A_{14} \Pi_4 \\ & D_2 & & \Pi_4 \\ & & D_3 & \Pi_4 \\ P_4 \tilde{A}_{41} \Pi_1 & P_4 \hat{A}_{42} \Pi_2 & P_4 \tilde{A}_{43} \Pi_3 & D_4 \end{pmatrix}$$

→ HBS : Hierarchically Block separable

Block-Separable Matrices

A *block-separable matrix* looks like this:

$$A = \begin{pmatrix} D_1 & P_1\tilde{A}_{12}\Pi_2 & P_1\tilde{A}_{13}\Pi_3 & P_1\tilde{A}_{14}\Pi_4 \\ P_2\tilde{A}_{21}\Pi_1 & D_2 & P_2\tilde{A}_{23}\Pi_3 & P_2\tilde{A}_{24}\Pi_4 \\ P_3\tilde{A}_{31}\Pi_1 & P_3\tilde{A}_{32}\Pi_2 & D_3 & P_3\tilde{A}_{34}\Pi_4 \\ P_4\tilde{A}_{41}\Pi_1 & P_4\tilde{A}_{42}\Pi_2 & P_4\tilde{A}_{43}\Pi_3 & D_4 \end{pmatrix}$$

Here:

- $\tilde{A}_{ij}$ smaller than A_{ij}
- D_i has full rank (not necessarily diagonal)
- P_i shared for entire row
- Π_i shared for entire column

Q: Why is it called that?

Engineering a cheap solve

Use the following notation:

$$B = \begin{pmatrix} 0 & P_1\tilde{A}_{12} & P_1\tilde{A}_{13} & P_1\tilde{A}_{14} \\ P_2\tilde{A}_{21} & 0 & P_2\tilde{A}_{23} & P_2\tilde{A}_{24} \\ P_3\tilde{A}_{31} & P_3\tilde{A}_{32} & 0 & P_3\tilde{A}_{34} \\ P_4\tilde{A}_{41} & P_4\tilde{A}_{42} & P_4\tilde{A}_{43} & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{pmatrix}, \quad \Pi = \begin{pmatrix} \Pi_1 & & & \\ & \Pi_2 & & \\ & & \Pi_3 & \\ & & & \Pi_4 \end{pmatrix}.$$

Then $A = D + B\Pi$ and

$$\begin{pmatrix} D & B \\ -\Pi & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \tilde{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}$$

$\to Dx + B\Pi x$

$\rightsquigarrow \tilde{x} = \Pi x$

is equivalent to $A\mathbf{x} = \mathbf{b}$.

Q: What are the matrix sizes? The vector lengths of $\mathbf{x}$ and $\tilde{\mathbf{x}}$?
(Π : small $\times$ large)

$$\begin{pmatrix} \Pi D^{-1} D & \Pi D^{-1} B \\ -\Pi & \mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \Pi D^{-1} b \\ 0 \end{pmatrix} \quad \big| \cdot \Pi D^{-1}$$

$$\left(\mathrm{Id} + \Pi D^{-1} B\right) \tilde{x} = \Pi D^{-1} b$$

All non zero entries of $\Pi D^{-1} B$ look like

$$\Pi_i D_i^{-1} P_i \tilde{A}_{ij}$$

$$\begin{pmatrix} & \tilde{A}_{12} \ \tilde{A}_{13} \\ & \\ & \end{pmatrix}$$

$$\tilde{A}_{ii} = \left(\Pi_i D_i^{-1} P_i\right)^{-1}$$

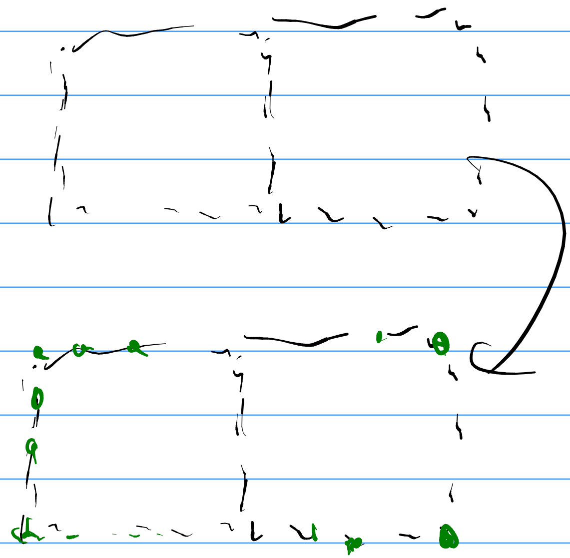
$$\begin{pmatrix} \tilde{A}_{11} & & \\ & \tilde{A}_{22} & \\ & & \ddots \end{pmatrix} (Id + \Pi D^{-1} B)\tilde{x} = \begin{pmatrix} \tilde{A}_{11} & & \\ & \ddots & \\ & & \tilde{A}_{44} \end{pmatrix} \Pi D^{-1} b$$

$$\begin{pmatrix} \tilde{A}_{11} & \hat{A}_{12} & \cdots \\ & \tilde{A}_{22} & \\ \vdots & & \end{pmatrix} \tilde{x} = \begin{pmatrix} \tilde{A}_{11} & & \\ & \ddots & \\ & & \tilde{A}_{44} \end{pmatrix} \Pi D^{-1} b$$

Uncompressed solve cost: $(4m)^3$

Compressed $(4K)^3$

$$\left(\frac{K}{m}\right)^3$$



Solving with Block-Separable Matrices

In order to get $O(N)$ complexity, could we apply this procedure recursively?

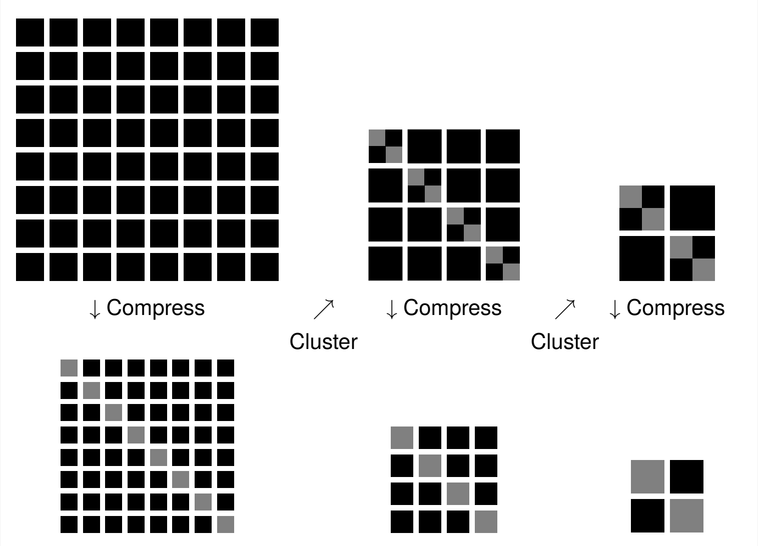


Figure credit: G. Martinsson, Boulder

Telescoping Factorization

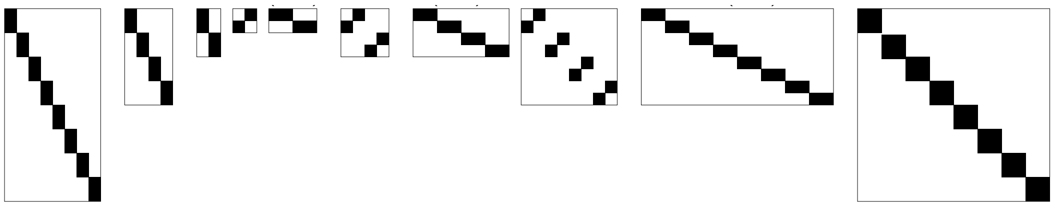
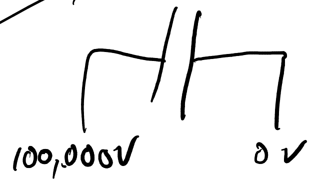


Figure credit: G. Martinsson, Boulder

Observations?

5 Outlook: Building a Fast PDE Solver

PDEs: Simple Ones First, More Complicated Ones Later



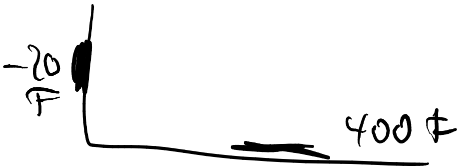
Laplace

$\triangle u = 0$

- Steady-state $\partial_t u = 0$ of wave propagation, heat conduction
- Electric potential u for applied voltage
- Minimal surfaces/"soap films"
- ∇u as velocity of incompressible flow

Heat:

$$\partial_t u = \Delta u$$



Helmholtz

$\triangle u + k^2 u = 0$

- Assume time-harmonic behavior $\tilde{u} = e^{\pm i\omega t} u(x)$ in time-domain wave equation:

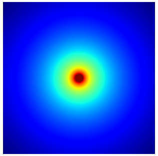
$$\partial_t^2 \tilde{u} = \triangle \tilde{u}$$

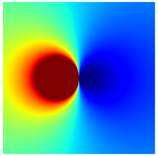
- Sign in $\tilde{u}$ determines direction of wave: Incoming/outgoing if free-space problem
- *Applications:* Propagation of sound, electromagnetic waves

Fundamental Solutions

Laplace

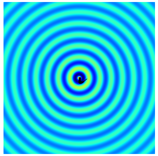
$-\triangle u = \delta$



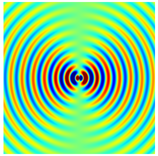


Helmholtz

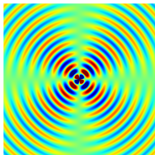
$\triangle u + k^2 u = \delta$



Monopole



Dipole



Quadrupole

$\Delta u = f$

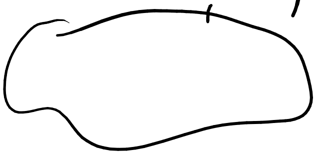
$\frac{e^{ikr}}{r}$

$H_0^{(1)}(kr)$

$\int -\Delta u \varphi = \int \delta \varphi$

$\int - \ldots = \varphi(0)$

$G(x,y)$



aka. *Free space Green's Functions*

$\int G(x,y)\, g(y)\, dy$ = solution to the BVP

How do you assign a precise meaning to the statement with the δ-function?

Why care about Green's functions?

What is a non-free-space Green's function? I.e. one for a specific domain?

Why not just use domain Green's functions?

What if we don't know a Green's function for our PDE... at all?

Fundamental solutions

Laplace

$$G(x) = \begin{cases} \frac{1}{-2\pi} \log |x| & \text{2D} \\ \frac{1}{4\pi} \frac{1}{|x|} & \text{3D} \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

Helmholtz

Monopole

$$G(x) = \begin{cases} \frac{i}{4} H_0^1(k|x|) & \text{2D} \\ \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} & \text{3D} \end{cases}$$

Dipole

$$\frac{\partial}{\partial_x} G(x)$$

Layer Potentials

$$(S_k\sigma)(x) := \int_\Gamma G_k(x-y)\sigma(y)ds_y$$

$$(S_k'\sigma)(x) := n\cdot\nabla_x PV\int_\Gamma G_k(x-y)\sigma(y)ds_y$$

$$(D_k\sigma)(x) := PV\int_\Gamma n\cdot\nabla_y G_k(x-y)\sigma(y)ds_y$$

$$(D_k'\sigma)(x) := n\cdot\nabla_x f.p.\int_\Gamma n\cdot\nabla_y G_k(x-y)\sigma(y)ds_y$$

- G_k is the Helmholtz kernel ($k=0\rightarrow$ Laplace)
- Operators–map function σ on Γ to...
  - ...function on $\mathbb{R}^n$
  - ...function on Γ (in particular)
- Alternate ("standard") nomenclature:

Ours	Theirs
S	V
D	K
S'	K'
D'	T

- S'' (and higher) analogously
- Called *layer potentials*:
  - S is called the *single-layer potential*
  - D is called the *double-layer potential*
- (Show pictures using `pytential/examples/layerpot.py`, observe continuity properties.)

How does this actually solve a PDE?

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u|_\Gamma = f|_\Gamma.$$

1. Pick *representation*:
$$u(x) := (S\sigma)(x)$$
2. Take (interior) limit onto Γ:
$$u|_\Gamma = S\sigma$$
3. Enforce BC:
$$u|_\Gamma = f$$
4. Solve resulting linear system:
$$S\sigma = f$$
(quickly–using the methods we've developed: It is precisely of the form that suits our fast algorithms!)
5. Obtain PDE solution in Ω by evaluating representation