TODAY

HW4 out

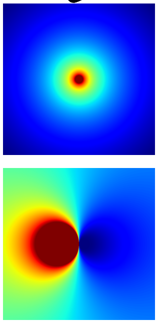
Project proposals

$$A\vec{x} = \vec{b}$$

Fundamental Solutions

Laplace

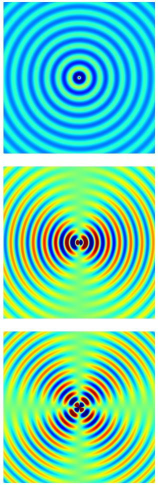
$-\triangle u = \delta$



$\Delta u = f$

Helmholtz

$\triangle u + k^2 u = \delta$



$\Delta u = f$

Monopole

$u = g$

Dipole

$u = G * f + \tilde{u}$

$\Delta(G * f) = (\Delta G) * f = \delta * f = f$

$\Delta(G * f + \tilde{u}) = f$

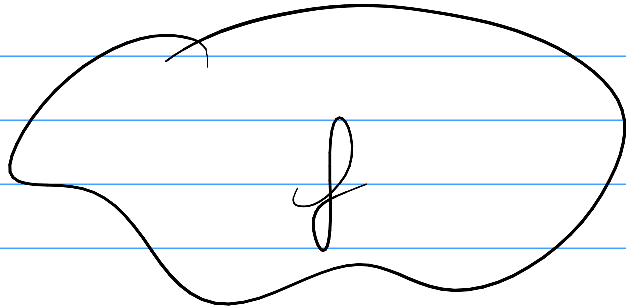
Quadrupole

$\Leftrightarrow \Delta \tilde{u} = 0$

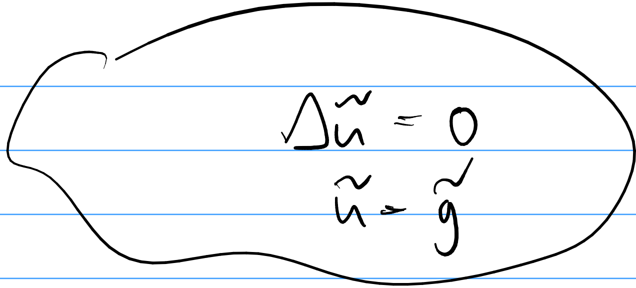
$\tilde{u} = g - G * f$

aka. *Free space Green's Functions*

> How do you assign a precise meaning to the statement with the δ-function?



$$(G * \rho)(x) = \int_{\Omega} G(x-y)\,\rho(y)\,dy$$



problem: do not know DOF

$$\tilde{u} = G^{?} * \tilde{g}$$

↑ domain Green's function

Attempt 2:

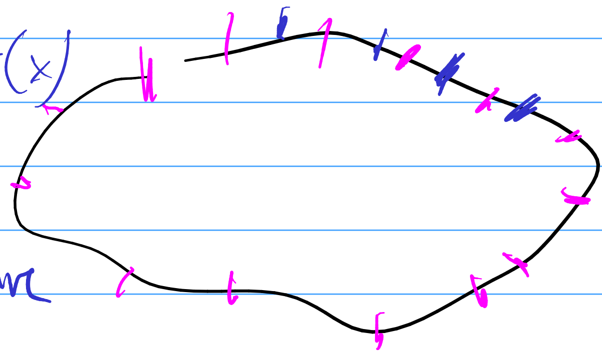
$$\tilde{u}(x) = G * \sigma$$

$$= \int_\Gamma G(x-y)\,\sigma(y)\,dy$$

① find σ

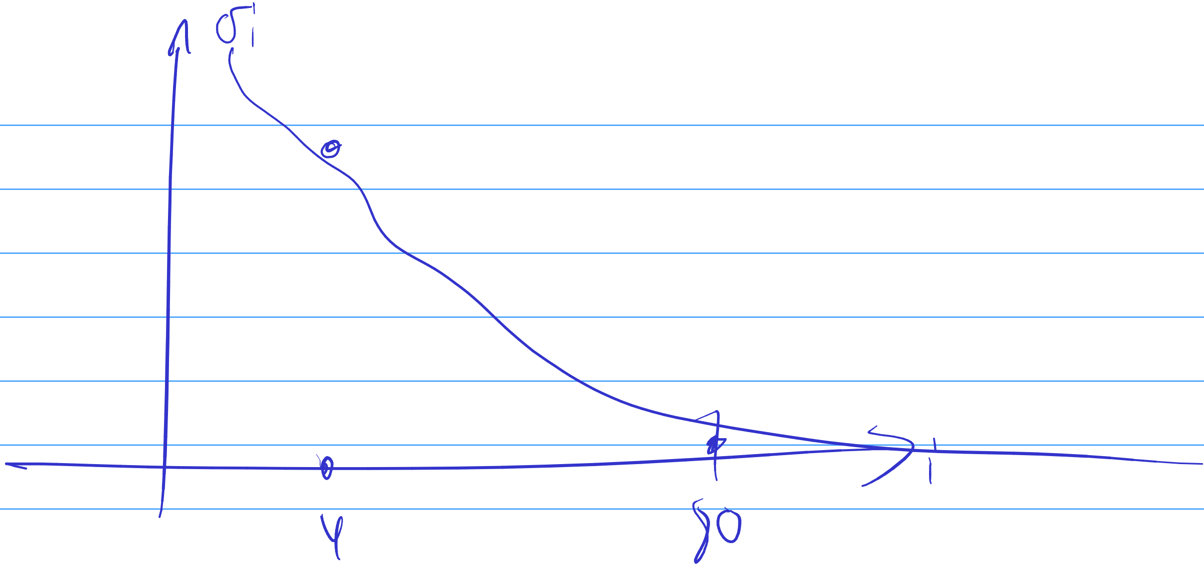
② Evaluate "representation"

$$\Delta \tilde{u} = 0 \quad \left| \quad \begin{aligned} &\lim_{x \to \partial\Omega^-} \tilde{u}(x) = \tilde{g}(x) \\ &\lim_{x \to \partial\Omega^-} \int_\Gamma G(x-y)\,\sigma(y)\,dy = \tilde{g}(x) \\ &\lim_{\dots} \underbrace{\int_\Gamma \log|x-y|\,\sigma(y)\,dy}_{\text{SLP: } S\sigma(x)} - \tilde{g}(x) \end{aligned} \right.$$



Questions:

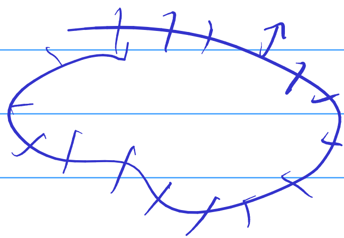
- Computational expense
- Conditioning / eigenvalue structure
- Existence / uniqueness
- Quadratures
- Discrete existence / uniqueness / error

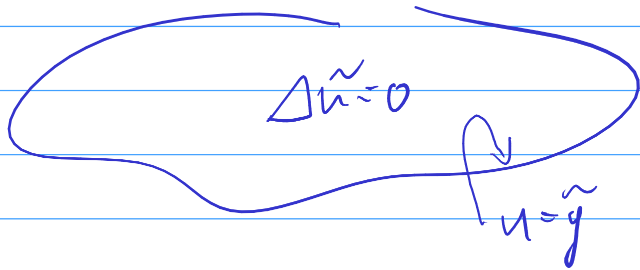


Instead of $\tilde{u} = S\sigma$; $\tilde{u}(x) = \int_{\Gamma} \frac{\partial G(x,y)}{\partial n(y)} \sigma(y)\, dy$

DLP

λ





$$\tilde{u} = D\sigma$$

$$\Delta(\partial_x G) = \partial_x(\Delta G) = \partial_x 0 = 0$$

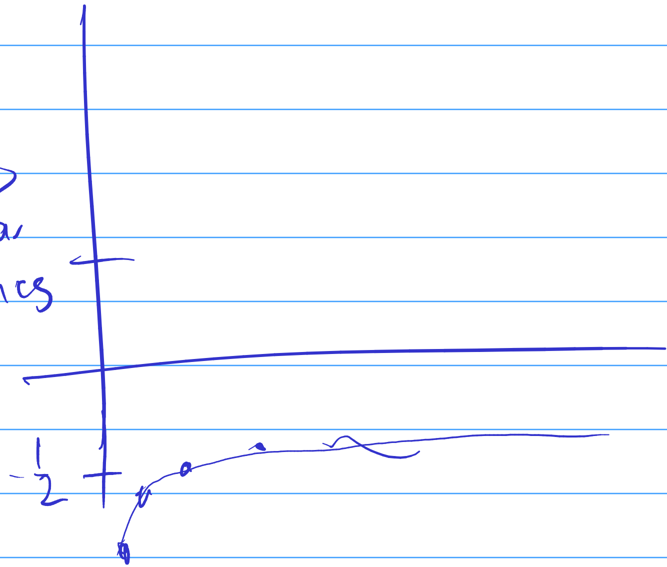
$$\tilde{g}(x) = \lim_{x \to \partial\Omega^-} D\sigma(x) = \pm\frac{\sigma}{2} + \text{"PV"}\, D\sigma(y)$$

on-surface

second kind IE

$$\left(-\frac{I}{2}+D\right)\sigma=\tilde{g}$$

plot singular values



Why care about Green's functions?

What is a non-free-space Green's function? I.e. one for a specific domain?

Why not just use domain Green's functions?

What if we don't know a Green's function for our PDE... at all?

Fundamental solutions

Laplace

$$G(x) = \begin{cases} \frac{1}{-2\pi} \log|x| & \text{2D} \\ \frac{1}{4\pi}\frac{1}{|x|} & \text{3D} \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

Helmholtz

$$G(x) = \begin{cases} \frac{i}{4} H_0^1(k|x|) & \text{2D} \\ \frac{1}{4\pi}\frac{e^{ik|x|}}{|x|} & \text{3D} \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

Monopole

Dipole

Layer Potentials

$$
\begin{aligned}
(S_k\sigma)(x) &:= \int_\Gamma G_k(x-y)\sigma(y)ds_y \\
(S_k'\sigma)(x) &:= n\cdot\nabla_x PV \int_\Gamma G_k(x-y)\sigma(y)ds_y \\
(D_k\sigma)(x) &:= PV \int_\Gamma n\cdot\nabla_y G_k(x-y)\sigma(y)ds_y \\
(D_k'\sigma)(x) &:= n\cdot\nabla_x f.p. \int_\Gamma n\cdot\nabla_y G_k(x-y)\sigma(y)ds_y
\end{aligned}
$$

- G_k is the Helmholtz kernel ($k = 0 \rightarrow$ Laplace)
- Operators–map function σ on Γ to...
  - ...function on $\mathbb{R}^n$
  - ...function on Γ (in particular)
- Alternate ("standard") nomenclature:

Ours	Theirs
S	V
D	K
S'	K'
D'	T

- S'' (and higher) analogously
- Called *layer potentials*:
  - S is called the *single-layer potential*
  - D is called the *double-layer potential*
- (Show pictures using `pytential/examples/layerpot.py`, observe continuity properties.)

How does this actually solve a PDE?

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u|_\Gamma = f|_\Gamma.$$

1. Pick *representation*:
$$u(x) := (S\sigma)(x)$$
2. Take (interior) limit onto Γ:
$$u|_\Gamma = S\sigma$$
3. Enforce BC:
$$u|_\Gamma = f$$
4. Solve resulting linear system:
$$S\sigma = f$$
(quickly–using the methods we've developed: It is precisely of the form that suits our fast algorithms!)
5. Obtain PDE solution in Ω by evaluating representation

Observations:

- One can choose representations relatively freely. Only constraints:
  - Can I get to the solution with this representation?

    I.e. is the solution I'm looking for represented?
  - Is the resulting integral equation solvable?

  Q: How would we know?
- Some representations lead to better integral equations than others. The one above is actually terrible (both theoretically and practically).

  Fix above: Use $u(x) = D\sigma(x)$ instead of $u(x) = S\sigma(x)$.

  Q: How do you tell a good representation from a bad one?
- Need to actually *evaluate* $S\sigma(x)$ or $D\sigma(x)$...

  Q: How?

$\rightarrow$ Need some theory

6 Going Infinite: Integral Operators and Functional Analysis

6.1 Norms and Operators

Norms

Definition 1 *(Norm) A* norm $\|\cdot\|$ *maps an element of a* vector space *into* $[0, \infty)$. *It satisfies:*

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality)*

Can create norm from *inner product*: $\|x\| = \sqrt{\langle x, x \rangle}$

Function Spaces

Name some function spaces with their norms.

Convergence

Name some ways in which a sequence can 'converge'.

Operators

X, Y: Banach spaces

$A : X \to Y$ linear operator

Definition 2 (Operator norm) $\|A\| := \sup\{\|Ax\| : x \in X, \|x\| = 1\}$

Theorem 1 *$\|A\|$ bounded $\Leftrightarrow$ A continuous*

Operators: Examples

Which of these is bounded as an operator on functions on the real line?

- Multiplication by a scalar
- "Left shift"
- Fourier transform
- Differentiation
- Integration
- Integral operators