TODAY:

- Functional Analysis
  - ↳ compact operators

HW2 graded

$$(A\vec{f})_i = \vec{u}\,\vec{v}^T \vec{f}$$

$$Af(x) = u(\vec{x}) \int_{\mathbb{R}^2} v(\vec{y})\, f(y)\, dy$$

$$A: \; ~~C(\mathbb{R}^2)~~ \to C(\mathbb{R}^2)$$

no Function spaces?

"left precon"

$MAx = Mb$

$Ax = b$

$AMy = b$

$$u(x) = D\sigma(x)$$

$$g(y) = \lim_{x \to \Gamma^+} u(x)$$

$$= +\tfrac{1}{2}\sigma + D\sigma(y)$$

$$S\sigma(x) = \int_\Gamma G(x,y)\,\sigma(y)\,dy$$

$$D\sigma(x) = \int_\Gamma \frac{\partial}{\partial n_y} G(x,y)\,\sigma(y)\,dy$$

Layer potentials as operators from the curve onto the curve; domain / range?

Integral equations: solvability?

↳ compact operators

↳ Fredholm alternative (for existence?)

Potential theory

↳ jump relations

↳ uniqueness

$\Delta u + k^2 u = 0$

How does this actually solve a PDE?

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u|_\Gamma = f|_\Gamma.$$

1. Pick *representation*:
$$u(x) := (S\sigma)(x)$$
2. Take (interior) limit onto Γ:
$$u|_\Gamma = S\sigma$$
3. Enforce BC:
$$u|_\Gamma = f$$
4. Solve resulting linear system:
$$S\sigma = f$$
(quickly–using the methods we've developed: It is precisely of the form that suits our fast algorithms!)
5. Obtain PDE solution in Ω by evaluating representation

Observations:

- One can choose representations relatively freely. Only constraints:
  - Can I get to the solution with this representation?

    I.e. is the solution I'm looking for represented?
  - Is the resulting integral equation solvable?

  Q: How would we know?
- Some representations lead to better integral equations than others. The one above is actually terrible (both theoretically and practically).

  Fix above: Use $u(x) = D\sigma(x)$ instead of $u(x) = S\sigma(x)$.

  Q: How do you tell a good representation from a bad one?
- Need to actually *evaluate* $S\sigma(x)$ or $D\sigma(x)$...

  Q: How?

→ Need some theory

6 Going Infinite: Integral Operators and Functional Analysis

6.1 Norms and Operators

Norms

Definition 1 *(Norm) A* norm $\|\cdot\|$ *maps an element of a* vector space *into* $[0, \infty)$*.*
It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality)*

Can create norm from *inner product*: $\|x\| = \sqrt{\langle x, x \rangle}$

Function Spaces

Name some function spaces with their norms.

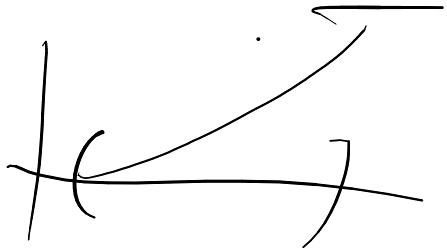
$L_2(\Omega)$; $\|f\|_2 = \sqrt{\int_\Omega |f|^2 \, dx}$

$L_p(\Omega)$ $\|f\|_p = \sqrt[p]{\int |f|^p \, dx}$

1

∞ $\|f\|_\infty = \sup |f|$

C^0 ; cont. and $\|f\|_\infty < \infty$



Convergence

Name some ways in which a sequence can 'converge'.

$f_1 \quad f_2 \quad \dots$

$f_1, \dots f_n, \dots \longrightarrow f \qquad (f_i)$ converge to f:

$\|f_i - f\| \to 0$

$\forall \varepsilon > 0 \ \exists n_0 \forall n \geq n_0 : \|f_n - f\| < \varepsilon$

Cauchy sequence:

$\forall \varepsilon > 0 \quad \exists n_0 \ \forall n, m \geq n_0 : \|f_n - f_m\| < \varepsilon$

$3., \ 3.1, \ 3.14, 3.141 \in \mathbb{Q}$

Completeness / Banach space: every Cauchy sequence has a limit

Operators

$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$

X, Y: Banach spaces

$A : X \to Y$ linear operator

Definition 2 (Operator norm) $\|A\| := \sup\{\|Ax\| : x \in X, \|x\| = 1\}$

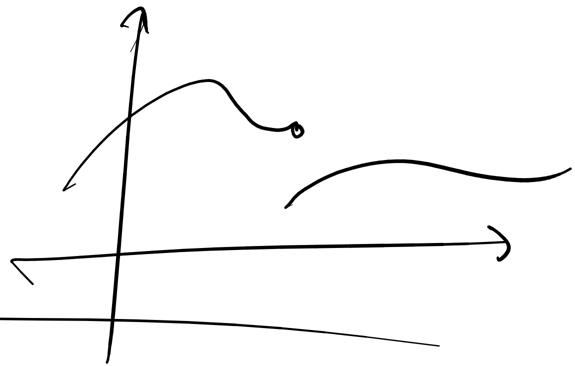
Theorem 1 $\|A\|$ *bounded* $\Leftrightarrow$ *A continuous*

$Ax = y$

$A(x+\varepsilon)$ "near" $Ax = y$

f continuous at x:

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x' : |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$



Operators: Examples

Which of these is bounded as an operator on functions on the real line?

- Multiplication by a scalar
- “Left shift”
- Fourier transform
- Differentiation
- Integration
- Integral operators

Integral Operators: Zoology

Volterra	Fredholm
$\int_a^x k(x,y)f(y)dy = g(x)$	$\int_G k(x,y)f(y)dy = g(x)$

First kind	Second kind
$\int_G k(x,y)f(y)dy = g(x)$	$f(x) + \int_G k(x,y)f(y)dy = g(x)$

Connections to Complex Variables

Complex analysis is *full* of integral operators:

- Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z-a} f(z)\, dz$$

- Cauchy's differentiation formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{1}{(z-a)^{n+1}} f(z)\, dz$$

Integral Operators: Boundedness (=Continuity)

Theorem 2 (Continuous kernel ⇒ bounded) *$G \subset \mathbb{R}^n$ closed, bounded ("compact"), $K \in C(G^2)$. Let*

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

Then

$$\|A\|_\infty = \max_{x\in G} \int_G |K(x,y)|dy.$$

Show '$\leqslant$'.

$$(I - A) f = g$$

$$f = (I - A)^{-1} g$$

$$\frac{1}{1-\alpha} = \sum_{i=0}^{\infty} \alpha^i$$

Solving Integral Equations

Given

$$(A\phi)(x) := \int_G K(x,y)\varphi(y)dy,$$

are we allowed to ask for a solution of

$$(\mathrm{Id} + A)\varphi = g?$$

Attempt 1: The Neumann series

Want to solve

$$\varphi - A\varphi = (I - A)\varphi = g.$$

Formally:

$$\varphi = (I - A)^{-1} g.$$

What does that remind you of?

6.2 Compactness

Compact sets

Definition 3 (Precompact/Relatively compact) *$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in X*

Definition 4 (Compact/'Sequentially complete') *$M \subseteq X$ compact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in M*

- Precompact $\Rightarrow$ bounded
- Precompact $\Leftarrow$ bounded (finite dim. only!)

Counterexample?

Compact Operators

X, Y: Banach spaces

Definition 5 (Compact operator) $T : X \to Y$ *is* compact $:\Leftrightarrow$ $T($*bounded set*$)$ *is precompact.*

- T, S compact $\Rightarrow \alpha T + \beta S$ compact
- *One of* T, S compact $\Rightarrow S \circ T$ compact
- T_n all compact, $T_n \to T$ in operator norm $\Rightarrow T$ compact

Questions:

- Let $\dim T(X) < \infty$. Is T compact?
- Is the identity operator compact?