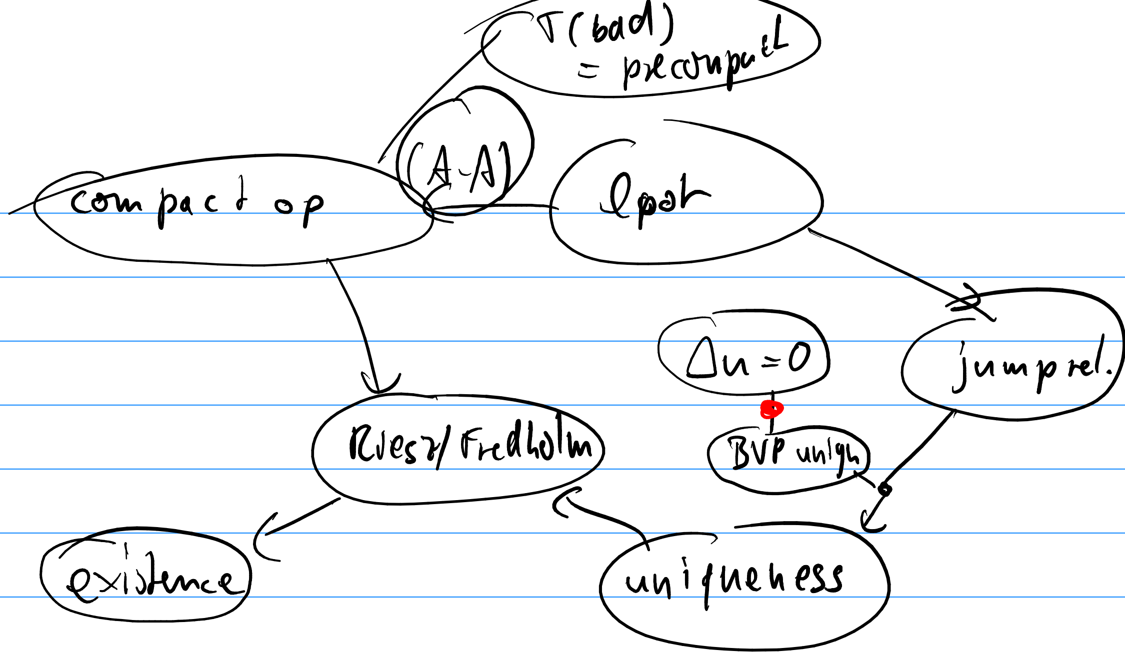
TODAY:

- project pres signup
- hw 3,4: grading
- hw5: coming



$$\|A\|_\infty \geq \left\| \int G(x,y)\, \varphi(y)\, dy \right\|$$

$I - \text{compact} : C(\Gamma) \to C(\Gamma)$

$(I - \text{compact})\,\varphi = g$

$$\lim_{x \to \partial\Omega +} D\varphi - \lim_{x \to \partial\Omega -} D\varphi = \varphi$$

compact:

$$\dim(N(I-A)) < \infty$$

$$\varphi \in N(I-A)$$

$$A\varphi = \varphi$$

$0 \in \sigma(A)$:

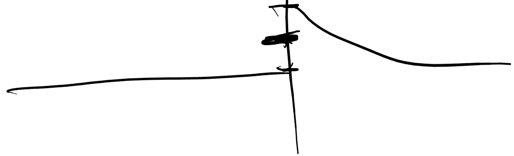
Suppose $0 \notin \sigma(A)$.

$\Rightarrow$ A is injective

$\underset{K3}{\Rightarrow}$ A is surjective

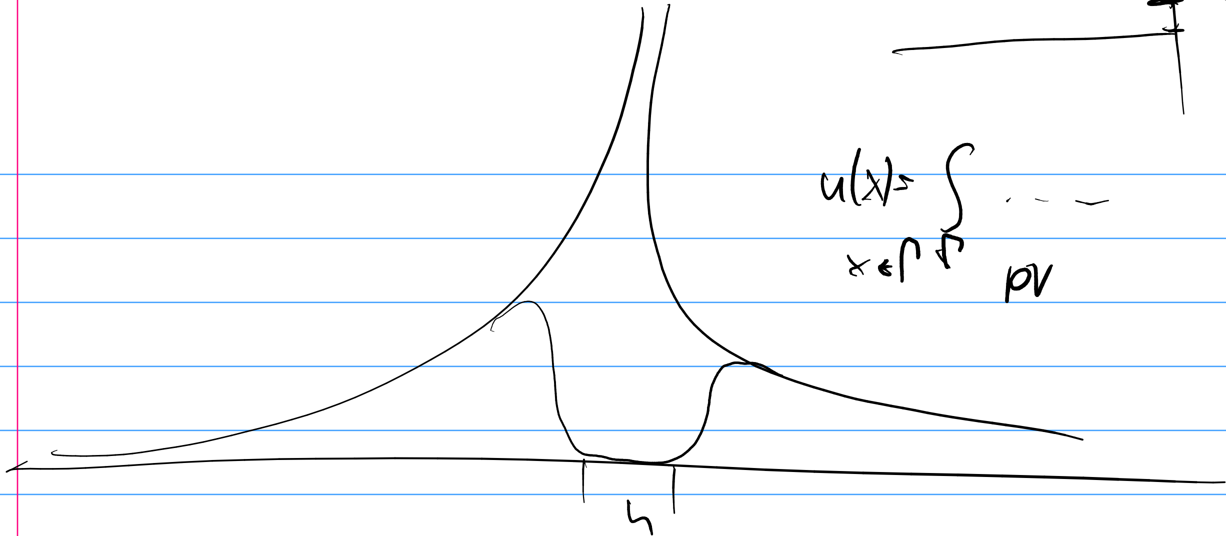
$\Rightarrow$ A^{-1} exists.

$\Rightarrow$ $I = AA^{-1}$ compact ↯

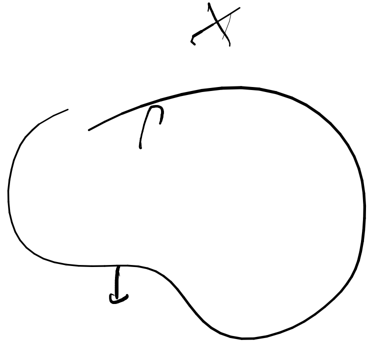


$$u(x) = \int_{x \in \Gamma} \dots$$

pv



Recap: Layer potentials



not us

$S\sigma$ $$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

$K'\sigma$ $$(S'\sigma)(x) := \mathrm{PV}\, \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

$K\sigma$ $$(D\sigma)(x) := \mathrm{PV} \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

$T\sigma$ $$(D'\sigma)(x) := \mathrm{f.p.}\, \hat{n}_{(x)}\cdot\nabla_x \int_\Gamma \hat{n}_{(y)}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

Important for us: Recover ‘average’ of interior and exterior limit without having to refer to off-surface values.

$$(A\varphi, \psi) = (\varphi, A^*\psi)$$

$$\int A\varphi(x)\,\psi(x)\,dx = \int \varphi(x)\,A^*\psi(x)\,dx$$

$$\int \int G(x,y)\varphi(y)\,dy\,\varphi(x)\,dx =$$

Green's Theorem

Theorem 11 (Green's Theorem [Kress LIE 2nd ed. Thm 6.3])

$$\int_\Omega u\triangle v + \nabla u \cdot \nabla v = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v)ds$$

$$\int_\Omega u\triangle v - v\triangle u = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u)ds$$

If $\triangle v = 0$, then
$u=1$

$$\int_{\partial\Omega} \hat{n}\cdot\nabla v = 0$$

What if $\triangle v = 0$ and $u = G(|y-x|)$ in Green's second identity?

Green's Formula

$$0 = \int_{\partial\Omega} (n\cdot\nabla v)\, ds$$

$$- \int v(y)\, \delta(y-x)\, dx = \int (\partial_n \psi) - D(v)$$

$$v(x) =$$

Theorem 12 (Green's Formula [Kress LIE 2nd ed. Thm 6.5]**)** *If* $\triangle u = 0$, *then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in \Omega \\ \frac{u(x)}{2} & x \in \partial\Omega \\ 0 & x \notin \bar{\Omega} \end{cases}$$

Suppose I know 'Cauchy data' $(u|_{\partial\Omega}, \hat{n} \cdot \nabla u|_{\partial\Omega})$ of u. What can I do?

What if Ω is an exterior domain?

What if $u = 1$? Do you see any practical uses of this?

$$-D(1) = \begin{cases} 1 & x \in \Omega \\ 0 & \text{on the exterior} \end{cases}$$

Things harmonic functions (don't) do

Theorem 13 (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7]) *If $\Delta u = 0$,*

$$u(x) = \overline{\int}_{B(x,r)} u(y)dy = \overline{\int}_{\partial B(x,r)} u(y)dy$$

Define $\overline{\int}$?

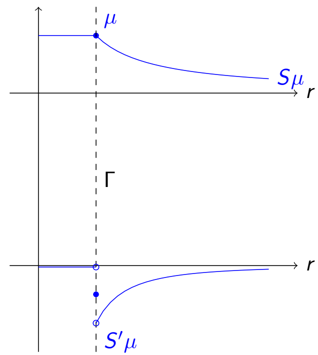
Trace back to Green's Formula (say, in 2D):

Theorem 14 (Maximum Principle [Kress LIE 2nd ed. 6.9]) *If $\triangle u = 0$ on compact set $\bar{\Omega}$:*
u attains its maximum on the boundary.

Suppose it were to attain its maximum somewhere inside an open set...

What do our *constructed* harmonic functions (i.e. layer potentials) do there?

Jump relations



Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".)

[Kress LIE 2nd ed. Thm. 6.14, 6.17,6.18]

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I \right) (\sigma)(x_0) \quad \Rightarrow \quad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I \right) (\sigma)(x_0) \quad \Rightarrow \quad [D\sigma] = \sigma$$

$$[D'\sigma] = 0$$

Truth in advertising: Assumptions on Γ?

Sketch the proof for the single layer.

Sketch proof for the double layer.

Green's Formula at Infinity (skipped)

$\Omega \subseteq \mathbb{R}^n$ bounded, C^1, connected boundary, $\triangle u = 0$, u bounded

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + (S_{\partial B_r}(\hat{n} \cdot \nabla u) - D_{\partial B_r} u)(x) = u(x)$$

for x between $\partial\Omega$ and B_r.

Now $r \to \infty$.

Behavior of individual terms?

Use mean value theorem and Gauss to estimate

$$|\nabla u| \leqslant C/r.$$

Theorem 15 (Green's Formula in the exterior [Kress LIE 2nd ed. Thm 6.10])

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + \text{PV} u_\infty = u(x)$$

for some constant u_∞. Only *for $n = 2$,*

$$u_\infty = \frac{1}{2\pi r} \int_{|y|=r} u(y) ds_y.$$

Theorem 16 (Green's Formula in the exterior [Kress LIE 2nd ed. Thm 6.10])

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + u_\infty = u(x)$$

Realize the power of this statement:

Can we use this to bound u as $x \to \infty$?
Consider the behavior of the fundamental solution as $r \to \infty$.

How about u's derivatives?

8 Boundary Value Problems

8.1 Laplace

Boundary Value Problems: Overview

	Dirichlet	**Neumann**
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ ⊕ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ ⓞ may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\|x\| \to \infty$ ⊕ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\|x\| \to \infty$ ⊕ unique

with $g \in C(\partial\Omega)$.

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

Dirichlet uniqueness: why?

Neumann uniqueness: why?

Truth in advertising: Missing assumptions on Ω?

What's a DtN map?

Next mission: Find IE representations for each.

Uniqueness of Integral Equation Solutions

Theorem 17 (Nullspaces [Kress LIE 2nd ed. Thm 6.20]**)**

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \operatorname{span}\{1\}$, $N(I/2 + S') = \operatorname{span}\{\psi\}$, *where* $\int \psi \neq 0$.

Show $N(I/2 - D) = \{0\}$.

Show $N(I/2 - S') = \{0\}$.

Show $N(I/2 + D) = \operatorname{span}\{1\}$.

What extra conditions on the RHS do we obtain?