TODAY

- hw5 ✓
- project sign-ups ✓
- proposals ✓

$$\left(\frac{1}{2} - 0\right)\varphi = 0 \Rightarrow \varphi = 0$$

$\rightarrow 1$

0

1

Theorem 12 (Green's Formula [Kress LIE 2nd ed. Thm 6.5]) *If $\triangle u = 0$, then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in \Omega \\ \frac{u(x)}{2} & x \in \partial\Omega \\ 0 & x \notin \Omega \end{cases}$$

Suppose I know 'Cauchy data' $(u|_{\partial\Omega}, \hat{n} \cdot \nabla u|_{\partial\Omega})$ of u. What can I do?

What if Ω is an exterior domain?

What if $u = 1$? Do you see any practical uses of this?

Things harmonic functions (don't) do

Theorem 13 (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7]) *If* $\Delta u = 0$,

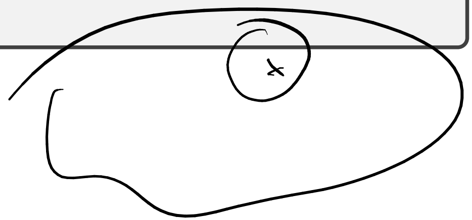
$$u(x) = \fint_{B(x,r)} u(y)dy = \fint_{\partial B(x,r)} u(y)dy$$

Define $\fint$?

$$\fint_\Omega f(x)\,dx = \frac{1}{|\Omega|}\int f(x)\,dx$$

Trace back to Green's Formula (say, in 2D):

Theorem 14 (Maximum Principle [Kress LIE 2nd ed. 6.9]) *If* $\triangle u = 0$ *on compact set* $\bar{\Omega}$*:*
u attains its maximum on the boundary.



$$u(x) = S(\hat{\vec{n}} \cdot \nabla u) - D(u)$$

$$= \frac{1}{2\pi} \int_{\partial B(x,r)} \log(|x-y|)\, n \cdot \nabla u(y)\, ds(y) - D(u)$$

$$= \frac{1}{2\pi} \log(r) \int_{\partial B(x,r)} \hat{n} \cdot \nabla u(y)\, ds(y) - D(u)$$

$$= -\frac{1}{2\pi} \int_{\partial B(x,r)} \hat{n}_y \cdot \nabla G(x-y)\, u(y)\, dy$$

$$\nabla \log(x) = \frac{x}{|x|^2}$$

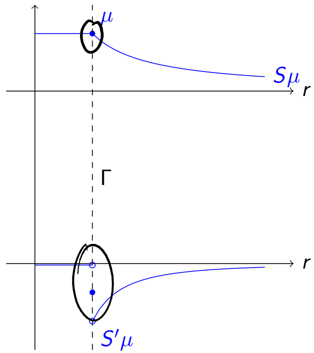
$$= -\frac{1}{2\pi} \int_{\partial B(x,r)} \frac{\vec{r}}{\|r\|} \frac{\vec{r}}{\|r\|^2}\, u(y)\, dy$$

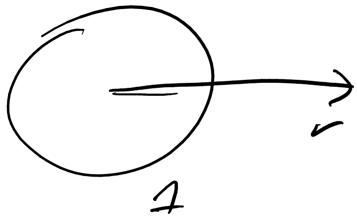
$$= -\frac{1}{2\pi r} \int_{\partial B(x,r)} u(y)\, dx$$

Suppose it were to attain its maximum somewhere inside an open set...

What do our *constructed* harmonic functions (i.e. layer potentials) do there?

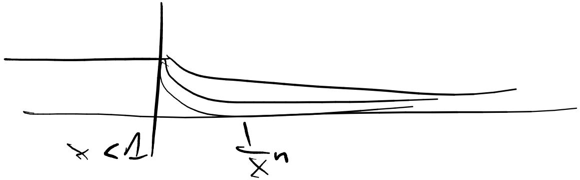
Jump relations





Let $[X] = X_+ - X_-$. (Normal points towards "$+$"="exterior".)

[Kress LIE 2nd ed. Thm. 6.14, 6.17,6.18]



$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I\right)(\sigma)(x_0) \quad \Rightarrow \quad [S'\sigma] = -\sigma$$

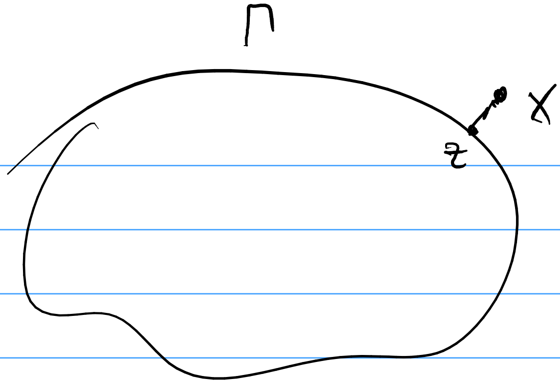
$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I\right)(\sigma)(x_0) \quad \Rightarrow \quad [D\sigma] = \sigma$$

$$[D'\sigma] = 0$$

Truth in advertising: Assumptions on Γ? $\partial\Omega$ is C^2

Sketch the proof for the single layer. (construct a sequence of functions with less of the singularity removed and show uniform convergence)

Sketch proof for the double layer.



Represent a target point near the bdry

$$\underset{\in \Gamma}{x = z} + h\,\hat{n}(z)$$

$$D\sigma(x) = \sigma(z)\,(D\,1)(x) + \underbrace{\big(D\sigma - D\sigma(z)\big)(x)}_{\displaystyle \int_\Gamma \hat{n}_y \cdot \nabla_y G(x,y)\,\big(\sigma(y) - \sigma(z)\big)\,ds(y)}$$

Green's Formula at Infinity (skipped)

$\Omega \subseteq \mathbb{R}^n$ bounded, C^1, connected boundary, $\triangle u = 0$, u bounded

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + (S_{\partial B_r}(\hat{n} \cdot \nabla u) - D_{\partial B_r} u)(x) = u(x)$$

for x between $\partial\Omega$ and B_r.

Now $r \to \infty$.

Behavior of individual terms?

Use mean value theorem and Gauss to estimate

$$|\nabla u| \leqslant C/r.$$

2D only: ("doesn't contribute") log $\frac{1}{r}$ / const ← longe range for u

Theorem 15 (Green's Formula in the exterior [Kress LIE 2nd ed. Thm 6.10])

$\int_{\partial\Omega} \hat{n}\cdot\nabla u = 0$

$$(S_{\partial\Omega}(\hat{n}\cdot\nabla u) - D_{\partial\Omega}u)(x) + \mathrm{PV}u_\infty = u(x)$$

for some constant u_∞. Only *for* $n = 2$,

$$u_\infty = \frac{1}{2\pi r}\int_{|y|=r} u(y)ds_y.$$

Theorem 16 (Green's Formula in the exterior [Kress LIE 2nd ed. Thm 6.10])

$$(S_{\partial\Omega}(\hat{n}\cdot\nabla u) - D_{\partial\Omega}u)(x) + u_\infty = u(x)$$

Realize the power of this statement:

Can we use this to bound u as $x \to \infty$?
Consider the behavior of the fundamental solution as $r \to \infty$.

How about u's derivatives?

$$\frac{1}{r^2}$$

8 Boundary Value Problems

8.1 Laplace

Boundary Value Problems: Overview

	Dirichlet	Neumann
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x\rvert \to \infty$ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x\rvert \to \infty$ unique

with $g \in C(\partial\Omega)$.

$f(x) = O(g(x))$ $\quad \frac{|f(x)|}{|g(x)|} \le C$

What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)

Dirichlet uniqueness: why? follows from the maximum principle

Neumann uniqueness: why?

Truth in advertising: Missing assumptions on Ω?

Neumann uniqueness:

u_1, u_2 harmonic $\quad \partial_n u_1 = \partial_n u_2 \quad \Big| \quad \tilde{u} = u_1 - u_2$

$$\int_\Omega \cancel{u \Delta u} + \nabla \tilde{u} \cdot \nabla \tilde{u} = \int_{\partial\Omega} \tilde{u} \, (\hat{n} \cdot \nabla \tilde{u}) \, ds$$

$\partial_n u_1 = \partial_n u_2$

$$\int_\Omega \|\nabla u\|_2^2 \qquad = 0$$

What's a DtN map?

Next mission: Find IE representations for each.

Uniqueness of Integral Equation Solutions

Theorem 17 (Nullspaces [Kress LIE 2nd ed. Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \text{span}\{1\}$, $N(I/2 + S') = \text{span}\{\psi\}$, *where* $\int \psi \neq 0$.

Show $N(I/2 - D) = \{0\}$.

Show $N(I/2 - S') = \{0\}$.

Show $N(I/2 + D) = \text{span}\{1\}$.

What extra conditions on the RHS do we obtain?

$\rightarrow$ "Clean" Existence for 3 out of 4.

Patching up Exterior Dirichlet (skipped)

Problem: $N(I/2 + S') = \{\psi\}$...but we do not know ψ.

Use a different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \qquad \to \qquad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in \Omega$)

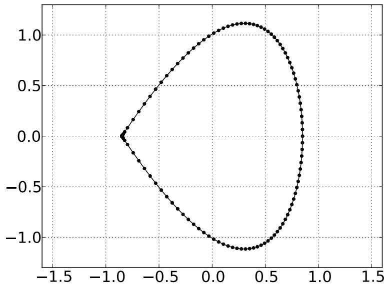
- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2} u(x) = ?$ on exterior
- Thus $\int \phi = 0$. Contribution of the second term?
- $\phi/2 + D\phi = 0$, i.e. $\phi \in N(I/2 + D) = ?$

- Existence/uniqueness?

$\rightarrow$ Existence for 4 out of 4.

Remaining key shortcoming of IE theory for BVPs?

Domains with Corners



What's the problem? *(Hint: Jump condition for constant density)*

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x,y)\phi(y) ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on} \pm \text{side} \rangle}{\pi} \phi$$

$\to$ non-continuous behavior of potential on Γ at x_0

What space have we been living in?

Fixes:

- I + Bounded (Neumann) + Compact (Fredholm)