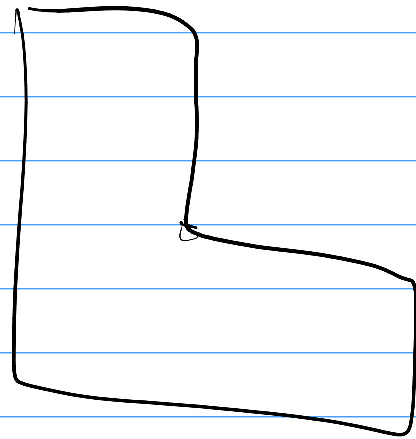
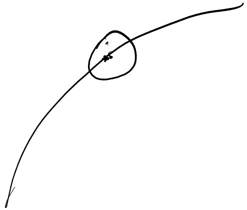
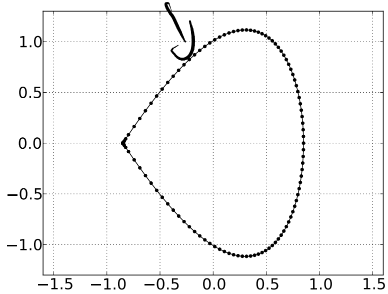
TODAY:

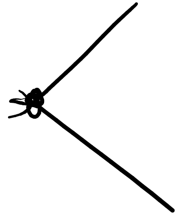
- Obdw: $\wedge$
- Helmholtz: the bad parts
- Calderón
- numerics



Domains with Corners







What's the problem? *(Hint: Jump condition for constant density)*

At corner x_0: (2D)

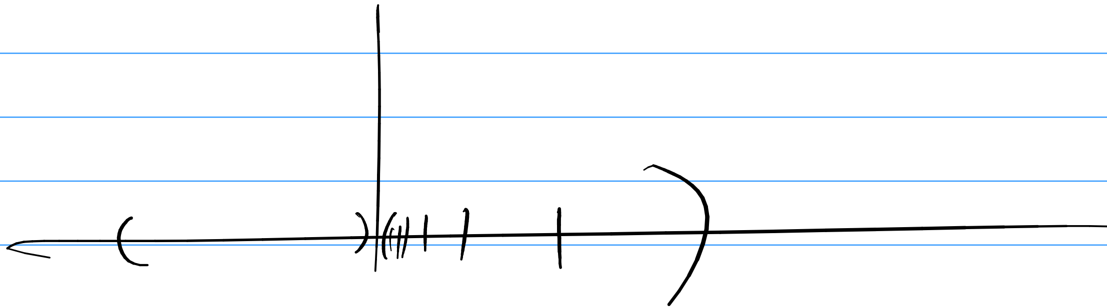
$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x,y)\phi(y)ds_y \pm \frac{1}{2}\frac{\langle \text{opening angle on } \pm \text{ side}\rangle}{\pi}\phi$$

$\rightarrow$ non-continuous behavior of potential on Γ at x_0

What space have we been living in?

Fixes:

- I + Bounded (Neumann) + Compact (Fredholm)



$$\frac{1}{x}$$

- Use L^2 theory

  (point behavior “invisible”)

Numerically: Needs consideration, but ultimately easy to fix.

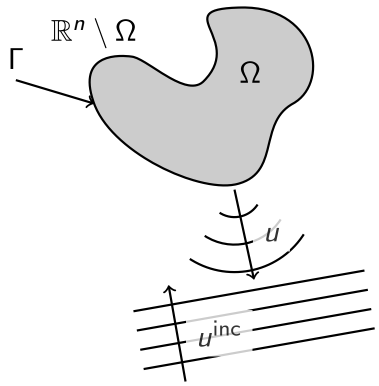
8.2 Helmholtz

Where does Helmholtz come from?

Derive the Helmholtz equation from the wave equation

$$\partial_t^2 U = c^2 \triangle U,$$

The prototypical Helmholtz BVP: A Scattering Problem



Ansatz:

$$u^{\mathrm{tot}} = u + u^{\mathrm{inc}}$$

Solve for scattered field u.

Helmholtz: Some Physics

Physical quantities:

- Velocity potential: $U(x, t) = u(x)e^{-i\omega t}$

  (fix phase by e.g. taking real part)
- Velocity: $v = (1/\rho_0)\nabla U$
- Pressure: $p = -\partial_t U = i\omega u e^{-i\omega t}$
  - Equation of state: $p = f(\rho)$

What's ρ_0?

What happens to a pressure BC as $\omega \to 0$?

Helmholtz: Boundary Conditions

- **Sound-soft**: Pressure remains constant
  - Scatterer "gives"
  - $u = f \rightarrow$ Dirichlet
- **Sound-hard**: Pressure same on both sides of interface
  - Scatterer "does not give"
  - $\hat{n} \cdot \nabla u = 0 \rightarrow$ Neumann
- **Impedance**: Some pressure translates into motion
  - Scatterer "resists"
  - $\hat{n} \cdot \nabla u + ik\lambda u = 0 \rightarrow$ Robin ($\lambda > 0$)
- **Sommerfeld** radiation condition: allow only outgoing waves

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) u(x) \rightarrow 0 \qquad (r \rightarrow \infty)$$

(where n is the number of space dimensions)

Many interesting BCs $\rightarrow$ many IEs! :)

Transmission between media: What's continuous?

Unchanged from Laplace

Theorem 18 (Green's Formula [Colton/Kress IAEST Thm 2.1]) *If $\triangle u + k^2 u = 0$, then*

$$(S(\hat{n}\cdot\nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

$H_0^{(1)}(k\cdot r)$

$$[Su] = 0$$

$$\lim_{x\to x_0\pm}(S'u) = \left(S' \mp \frac{1}{2}I\right)(u)(x_0) \quad \Rightarrow \quad [S'u] = -u$$

$$\lim_{x\to x_0\pm}(Du) = \left(D \pm \frac{1}{2}I\right)(u)(x_0) \quad \Rightarrow \quad [Du] = u$$

$$= 0$$

Why is singular behavior (esp. jump conditions) unchanged?

$$\frac{e^{ikr}}{r} = \frac{1}{r} + \frac{e^{ikr}-1}{r}$$

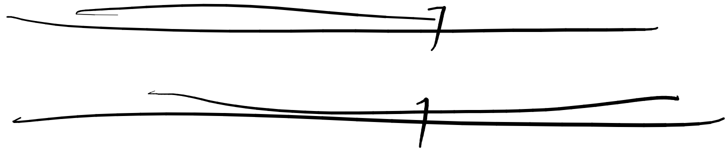
Why does Green's formula survive?
Remember Green's theorem:

$$\int_{\Omega} u\triangle v - v\triangle u = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u)ds$$

$\Delta u + k^2 u = 0$

Δ

$\Delta + k^2$



Resonances

$-\triangle$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.

What does that have to with Helmholtz?

Why could it cause grief?

Helmholtz: Boundary Value Problems

Find $u \in C(\bar{D})$ with $\triangle u + k^2 = 0$ such that

	Dirichlet		Neumann	
Int.	$\lim_{x \to \partial D-} u(x) = g$ unique ($-$resonances)	$\frac{I}{2} - D$	$\lim_{x \to \partial D-} \hat{n} \cdot \nabla u(x) = g$ unique ($-$resonances)	$\frac{I}{2} + S'$
Ext.	$\lim_{x \to \partial D+} u(x) = g$ Sommerfeld unique	$\frac{I}{2} + D$	$\lim_{x \to \partial D+} \hat{n} \cdot \nabla u(x) = g$ Sommerfeld unique	$\frac{I}{2} - S'$

with $g \in C(\partial D)$.

Find layer potential representations for each.

inherits nullspaces (because adj.)

spurious resonances

$$u(x) = S\sigma - iD(\sigma)$$

"CFIE"

Combined-field ext Neumann: $\left(\frac{I}{2} - S' - D'\right)\sigma = g$

$u(x) = S\sigma - i D S\sigma$

$D'S = (S')^2 - \frac{I}{4}$

Patching up resonances

Issue: Ext. IE inherits non-uniqueness from 'adjoint' int. BVP

Fix: Tweak representation [Brakhage/Werner '65, ...]

(also called the '*CFIE*'–'*combined field integral equation*')

$$u = D\phi - i\alpha S\phi$$

(α: tuning knob $\to$ 1 is fine, $\sim k$ better for large k)

How does this help?

Uniqueness for remaining IEs similar. (skipped)

8.3 Calderón identities

Show that D' is self adjoint.

Show that $(S\varphi, D'\psi) = ((S' + I/2)\varphi, (D - I/2)\psi)$.

$(\varphi, SD'\psi)$?

$$(D'\varphi, \psi) = (\varphi, D'\psi)$$

$$u := D\varphi \qquad v := D\psi$$

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} (n\cdot\nabla u)v - u(n\cdot\nabla v) = 0$$

$$(D'\varphi, \psi) = (D'\varphi, [v])$$
$$= (D'\varphi, v^+) - (D'\varphi, v^-)$$
$$= ((D\varphi)^+, v') - ((D\varphi)^-, v')$$
$$= ([u], v') = (\varphi, D'\psi)$$

$$SD'\psi = \left(D^2 - \frac{I}{4}\right)\psi$$

Calderón Identities: Summary

- $SD' = D^2 - I/4$
- $D'S = S'^2 - I/4$

Also valid for Laplace (jump relation same after all!)

Why do we care?

9 Back from Infinity: Discretization

9.1 Fundamentals: Meshes, Functions, and Approximation

Numerics: What do we need?

- Discretize curves and surfaces
  - Interpolation
  - Grid management
  - Adaptivity
- Discretize densities
- Discretize integral equations
  - Nyström, Collocation, Galerkin
- Compute integrals on them
  - "Smooth" quadrature
  - Singular quadrature
- Solve linear systems

$$Au = f$$

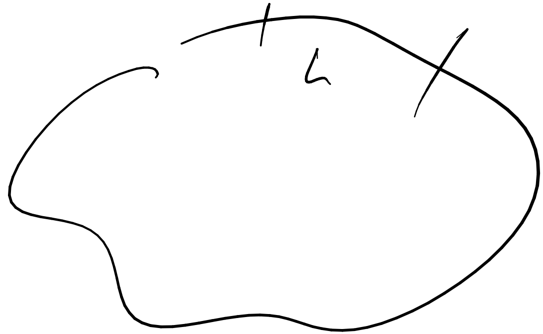
$$(Au, \varphi) = (f, \varphi)$$

Constructing Discrete Function Spaces

Floating point numbers (*Degrees of Freedom*/*DoFs*) $\rightarrow$ Functions

$\leftarrow$

$$\|f - f_h\| \leq C \cdot h$$



Discretization relies on three things:

- Base/reference domain
- Basis of functions
- Meaning of DoFs

Related finite element concept: *Ciarlet triple*

> Discretization options for a curve?

What do the DoFs mean?

Common DoF choices:

- Point values of function
- Point values of (directional?) derivatives
- Basis coefficients
- Moments

Often: useful to have both “modes”, “nodes”, jump back and forth

Why high order?

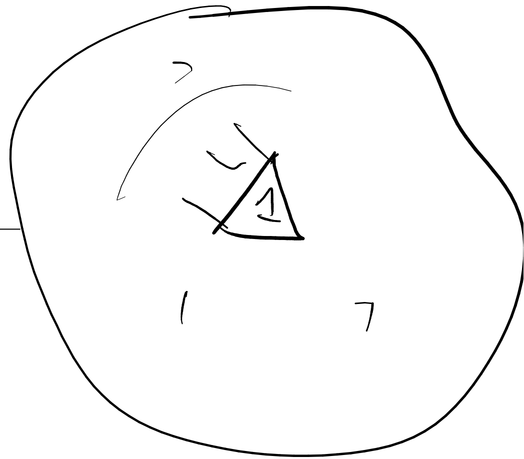
Order p: Error bounded as

$$|u_h - u| \leq Ch^p$$

Thought experiment:

First order	Fifth order
1,000 DoFs $\approx$ 1,000 triangles	1,000 DoFs $\approx$ 66 triangles
Error: 0.1	Error: 0.1
Error: 0.01 $\to$?	Error: 0.01 $\to$?

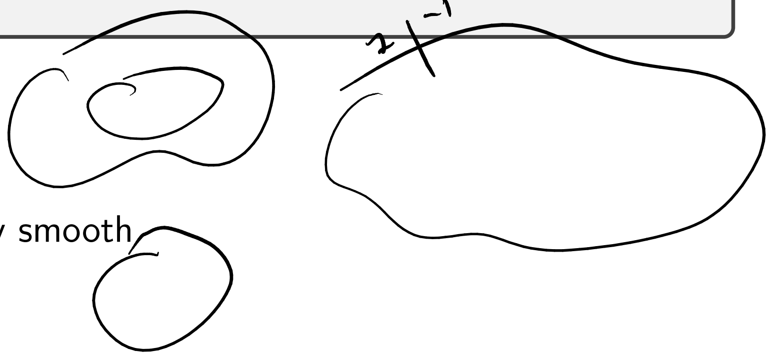
100,000 triangles



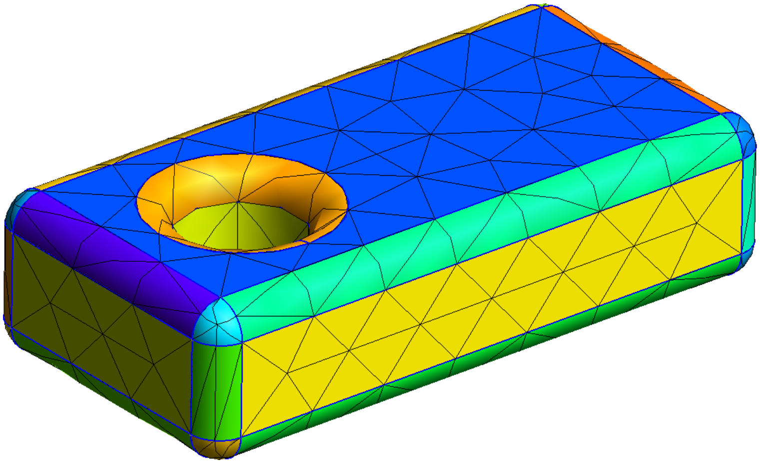
Complete the table.

Remarks:

- Want $p \geq 3$ available.
- **Assumption:** Solution sufficiently smooth
- Ideally: p chosen by user



What is an Unstructured Mesh?



Why have an unstructured mesh?

What is the trade-off in going unstructured?

Fixed-order vs Spectral

Fixed-order	Spectral
Number of DoFs n $\sim$ Number of 'elements'	Number of DoFs n $\sim$ Number of modes resolved
$$\text{Error} \sim \frac{1}{n^p}$$	$$\text{Error} \sim \frac{1}{C^n}$$
Examples? • Piecewise Polynomials	Examples? • Global Fourier • Global Orth. Polynomials

What assumptions are buried in each of these?

What should the DoFs be?

What's the difficulty with purely modal discretizations?

Vandermonde Matrices

$$\begin{pmatrix} x_0^0 & x_0^1 & \cdots & x_0^n \\ x_1^0 & x_1^1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = ?$$

Generalized Vandermonde Matrices

$$\begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = ?$$

Generalized Vandermonde Matrices

$$\begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{pmatrix} \text{MODAL COEFFS} = \text{NODAL COEFFS}$$

Node placement?

Vandermonde conditioning?

What about multiple dimensions?

Common Operations

(Generalized) Vandermonde matrices simplify common operations:

- Modal $\leftrightarrow$ Nodal ("Global interpolation")
  - Filtering
  - Up-/Oversampling
- Point interpolation (Hint: solve using V^T)
- Differentiation
- Indefinite Integration
- Inner product
- Definite integration