TODAY

- Nyström

- hw4 graded
- theory problem hw4.1 ←
- short office hour today

$$\Gamma - A_i \quad \begin{bmatrix} A_{11} & - & A_{1k} \\ & & \\ A_{k1} & & A_{kk} \end{bmatrix}$$

(A_{ik}) c $\Rightarrow A$ comp

A c $\Rightarrow (A_{ik})$ c

$$A : \underset{(X_1 \times \cdots \times X_n)}{X} \longrightarrow \underset{(X_1 \times \cdots \times X_n)}{X}$$

<u>Arzelà-Ascoli</u>

says wh a abounded subset of $C(\Omega)$ is compact.

Fixed-order vs Spectral

Fixed-order	Spectral
Number of DoFs n $\sim$ Number of 'elements' $$\text{Error} \sim \frac{1}{n^p}$$ Examples? • Piecewise Polynomials	Number of DoFs n $\sim$ Number of modes resolved $$\text{Error} \sim \frac{1}{C^n}$$ Examples? • Global Fourier • Global Orth. Polynomials

What assumptions are buried in each of these?

What should the DoFs be?

What's the difficulty with purely modal discretizations?

Vandermonde Matrices

$$\begin{pmatrix} x_0^0 & x_0^1 & \cdots & x_0^n \\ x_1^0 & x_1^1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = ?$$

Generalized Vandermonde Matrices

$$\begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = ?$$

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Node placement?

Vandermonde conditioning?

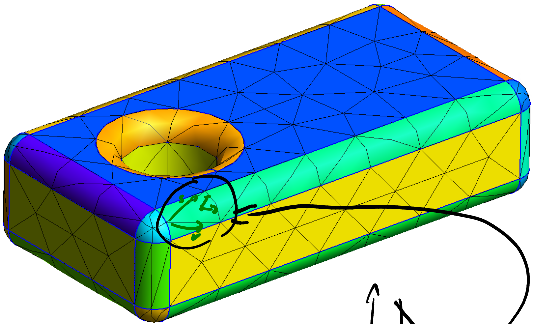
What about multiple dimensions?

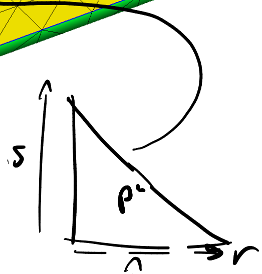
Common Operations

(Generalized) Vandermonde matrices simplify common operations:

- Modal $\leftrightarrow$ Nodal ("Global interpolation")
  - Filtering
  - Up-/Oversampling
- Point interpolation (Hint: solve using V^T)
- Differentiation
- Indefinite Integration
- Inner product
- Definite integration

Unstructured Mesh





- Design a data structure to represent this
- Compute normal vectors
- Compute area
- Compute integral of a function
- How is the function represented?

mappin

$\Psi(r,s) = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Psi = \begin{pmatrix} \sum \alpha_{1i} \varphi_i \\ \sum \alpha_{2i} \varphi_i \end{pmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix}$

∂_r, ∂_s

$\int_\Delta$

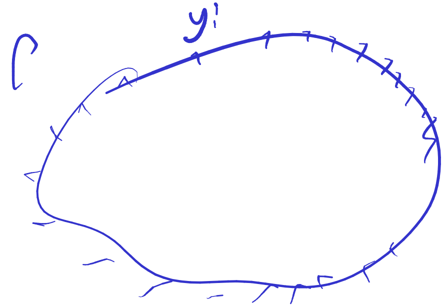
(φ_i)

Poly space of deg n

$|f - \Pi_n f| \leq C \cdot h^{n+1}$

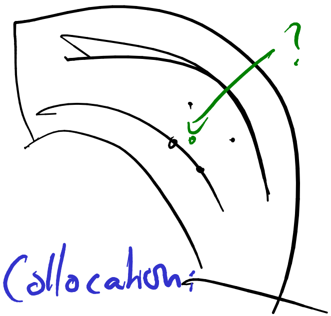
9.2 Integral Equation Discretizations

Integral Equation Discretizations: Overview



$$\phi(x) - \int_\Gamma K(x,y)\phi(y)dy = f(y)$$

Nyström	Projection
• Approximate integral by quadrature: $\int_\Gamma f(y)dy \to \sum_{k=1}^{n} \omega_k f(y_k)$ • Evaluate quadrature'd IE at quadrature nodes, solve	• Consider residual: $R := \phi - A\phi - f$ • Pick projection P_n onto finite-dimensional subspace $P_n\phi := \sum_{k=1}^{n} \langle \phi, v_k \rangle w_k$ $\to$ DOFs $\langle \phi, v_k \rangle$ • Solve $P_n R = 0$



- Equivalent to projection: Test IE with test functions
- Important in projection methods: *sub*space (e.g. of $C(\Gamma)$)

upside: clean math
downside: not implementable

Collocation:
$v_i = \delta(x - x_i)$
φ_i - poly
Galerkin:
$\varphi_i = v_i$ .. polynomials
Petrov-Galerkin
$\varphi_i \neq v_i$ .. = polynomials

Name some possible bases for projection?

Name some generic discrete projection bases.

Collocation and Nyström: the same?

Are projection methods implementable?

$A\varphi = f$

$(I-A)\varphi = f$ ← second-kind only!

Nyström discretizations

$$\varphi_n(x) - \sum_{k=1}^{n} \omega_k K(x, y_k)\varphi_n(y_k) = f(x)$$

Nyström consists of two distinct steps:

1. Approximate integral by quadrature:

$$\varphi_n(x) - \sum_{k=1}^{n} \omega_k K(x, y_k)\cancel{\varphi_n(y_k)} = f(x) \tag{1}$$

$\varphi_k^{(n)} \ldots$

2. Evaluate quadrature'd IE at quadrature nodes, solve discrete system

$$\varphi_j^{(n)} - \sum_{k=1}^{n} \omega_k K(x_j, y_k)\varphi_k^{(n)} = f(x_j) \tag{2}$$

with $x_j = y_j$ and $\varphi_j^{(n)} = \varphi_n(x_j) = \varphi_n(y_j)$

Is version (1) solvable?

What's special about (2)?

Solution density also only known at point values. But: can get approximate continuous density. How?

Assuming the IE comes from a BVP. Do we also only get the BVP solution at discrete points?

Does (1) $\Rightarrow$ (2) hold?

Does (2) $\Rightarrow$ (1) hold?

What good does that do us?

Does Nyström work for first-kind IEs?

$u(x) = \sum w_k K(x, y_k) \varphi_k$

Convergence for Nyström

$f_n \to f$

pointwise

uniform $\to$ norm conv. $\|\cdot\|_\infty$

Increase number of quadrature points n:

Get sequence (A_n)

Want $A_n \to A$ in some sense

What senses of convergence are there for sequences of functions f_n?

What senses of convergence are there for sequences of operators A_n?

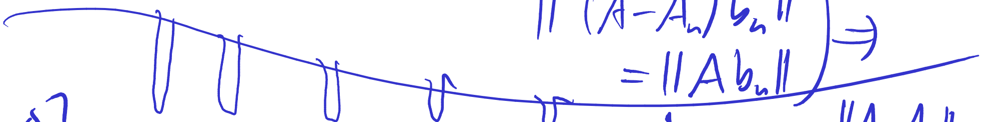
- for sequences of operators (A_n)?

Want norm convergence $\|A_n - A\|_\infty \to 0$

Will we get norm convergence $\|A_n - A\|_\infty \to 0$ for Nyström?

Is functionwise convergence good enough?

$\|(A_n - A)A\|_\infty \to 0$?



b_n

$\|(A-A_n)b_n\| = \|Ab_n\| \Rightarrow \|A-A_n\| \geq \|A\|$

Compactness-Based Convergence

X Banach space (think: of functions)

Theorem 19 (Not-quite-norm convergence [Kress LIE 2nd ed. Cor 10.4]) $A_n : X \to X$ *bounded linear operators,*
functionwise convergent to $A : X \to X$
Then convergence is uniform on compact subsets $U \subset X$*, i.e.*

$$\sup_{\phi \in U} \|A_n\phi - A\phi\| \to 0 \qquad (n \to \infty)$$

How is this different from norm convergence?

Set $\mathcal{A}$ of operators $A : X \to X$

Definition 15 (Collectively compact) $\mathcal{A}$ *is called* collectively compact *if and only if*
for $U \subset X$ *bounded,* $\mathcal{A}(U)$ *is relatively compact.*

What was relative compactness (=precompactness)?

Is each operator in the set $\mathcal{A}$ compact?

When is a sequence collectively compact?

Is the limit operator of such a sequence compact?

How can we use the two together?

Making use of Collective Compactness

X Banach space, $A_n : X \to X$, (A_n) collectively compact, $A_n \to A$ functionwise.

Corollary 1 (Post-compact convergence [Kress LIE 2nd ed. Cor 10.8]**)**

- $\|(A_n - A)A\| \to 0$
- $\|(A_n - A)A_n\| \to 0$

$(n \to \infty)$

Anselone's Theorem

Assume:

$(I - A)^{-1}$ exists, with $A : X \to X$ compact, $(A_n) : X \to X$ collectively compact and $A_n \to A$ functionwise.

Theorem 20 (Nyström error estimate [Kress LIE 2nd ed. Thm 10.9]) *For sufficiently large n, $(I - A_n)$ is invertible and*

$$\|\phi_n - \phi\| \leq C(\|(A_n - A)\phi\| + \|f_n - f\|)$$

$$C = \frac{1 + \|(I - A)^{-1} A_n\|}{1 - \|(I - A)^{-1}(A_n - A)A_n\|}$$

$I + (I - A)^{-1} A = ?$

Show the theorem.

Nyström: *specific to I+compact.* Why?

Nyström: Collective Compactness

Assume

$$\sum |\text{quad. weights for } n \text{ points}| \leq C \qquad \text{(independent of } n\text{)} \tag{3}$$

We've *assumed* collective compactness. Do we have that?

Also assumed functionwise uniform convergence, i.e. $\|A_n\phi - A\phi\| \to 0$ for each ϕ.

9.3 Integral Equation Discretizations: Projection

Error Estimates for Projection

X Banach spaces, $A : X \to X$ injective, $P_n : X \to X_n$

Theorem 21 (Céa's Lemma [Kress LIE 2nd ed. Thm 13.6]) *Convergence of the projection method*
$\Leftrightarrow$ *There exist n_0 and M such that for $n \geq n_0$*

1. *$P_n A : X_n \to X_n$ are invertible,*
2. *$\|(P_n A)^{-1} P_n A\| \leq M$.*

In this case,

$$\|\phi_n - \phi\| \leq (1 + M) \inf_{\psi \in X_n} \|\phi - \psi\|$$

Proof? (skipped)

Core message of the theorem?

What goes into P_n?

Note domain of invertibility for $P_n A$.

Domain/range of $(P_n A)^{-1} P_n A$?

Relationship to conditioning?

Relationship to second-kind?

Exact projection methods: hard. (Why?) What if we implement a perturbation? (i.e. apply quadature instead of computing exact integrals?)

Decisions, Decisions: Nyström or Galerkin?

Quote Kress LIE, 2nd ed., p. 244 (Sec. 14.1):

> [...] the Nyström method is generically stable whereas the collocation and Galerkin methods may suffer from instabilities due to a poor choice of basis for the approximating subspace.

Quote Kress LIE, 2nd ed., p. 244 (Sec. 13.5):

> In principle, for the Galerkin method for equations of the second kind the same remarks as for the collocation method apply. As long as numerical quadratures are available, in general, the Galerkin method cannot compete in efficiency with the Nyström method.
> Compared with the collocation method, it is less efficient, since its matrix elements require double integrations.

Need good quadratures to use Nyström.

Remaining advantage of Galerkin: