Today

- IE overview
- Loads of abstract math
  - $(I - A)\varphi = f$
  - compact operators

Announcements

- HW3
- Project Proposal

PDEs: Simple Ones First, More Complicated Ones Later

Laplace
$\triangle u = 0$

- Steady-state $\partial_t u = 0$ of wave propagation, heat conduction
- Electric potential u for applied voltage
- Minimal surfaces/"soap films"
- ∇u as velocity of incompressible flow

Helmholtz
$\triangle u + k^2 u = 0$

- Assume time-harmonic behavior $\tilde{u} = e^{\pm i\omega t} u(x)$ in time-domain wave equation:

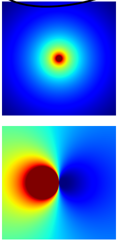
$$\partial_t^2 \tilde{u} = \triangle \tilde{u}$$

- Sign in $\tilde{u}$ determines direction of wave: Incoming/outgoing if free-space problem
- *Applications:* Propagation of sound, electromagnetic waves

Fundamental Solutions

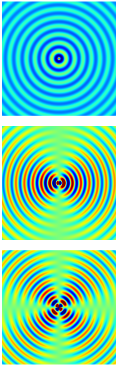
Laplace

$-\triangle u = \delta$



Helmholtz

$\triangle u + k^2 u = \delta$



aka. *Free space Green's Functions*

How do you assign a precise meaning to the statement with the δ-function?

$$\int \Delta u \varphi = \int \delta \varphi$$

Green's Functions

Why care about Green's functions?

$\Delta u = f$

$$\Delta G = \delta \quad \Rightarrow \quad \Delta(\overbrace{G * f}^{u}) = (\Delta G) * f = \delta * f = f$$

$$\partial_x (a * b) = (\partial_x a * b)$$

$$(G * f)(x) = \int_{\mathbb{R}^n} G(y) f(x-y)\, dy$$

What is a non-free-space Green's function? I.e. one for a specific domain?

$\Delta u = 0$ on Ω $\quad u = g$ on $\partial\Omega$

$G_d \leftarrow$ domain Green's function

$$u(x) = \int_{\partial\Omega} G_d(x-y)\, g(y)\, dS_y$$

Green's Functions (II)

Why not just use domain Green's functions?

They're unavailable.

What if we don't know a Green's function for our PDE... at all?

you can often still get away with a "near match".

Fundamental Solutions

Laplace

$$G(x) = \begin{cases} \frac{1}{-2\pi} \log |x| & 2D \\ \frac{1}{4\pi} \frac{1}{|x|} & 3D \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

Helmholtz

$$G(x) = \begin{cases} \frac{i}{4} H_0^1(k|x|) & 2D \\ \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} & 3D \end{cases}$$

$$\frac{\partial}{\partial_x} G(x)$$

Layer Potentials (I)

$\Gamma = \partial\Omega$

$$\rightarrow (S_k\sigma)(x) := \int_\Gamma G_k(x-y)\sigma(y)ds_y \longrightarrow \text{everywhere}$$

$$(S'_k\sigma)(x) := n\cdot\nabla_x PV\int_\Gamma G_k(x-y)\sigma(y)ds_y \longrightarrow \Gamma$$

$$\rightarrow (D_k\sigma)(x) := PV\int_\Gamma n\cdot\nabla_y G_k(x-y)\sigma(y)ds_y \longrightarrow \text{everywhere}$$

$$(D'_k\sigma)(x) := n\cdot\nabla_x f.p.\int_\Gamma n\cdot\nabla_y G_k(x-y)\sigma(y)ds_y \longrightarrow \Gamma$$

- G_k is the Helmholtz kernel ($k=0 \rightarrow$ Laplace)
- Operators–map function σ on Γ to...
  - ...function on $\mathbb{R}^n$
  - ...function on Γ (in particular)

Layer Potentials (II)

- Alternate ("standard") nomenclature:

Ours	Theirs
S	V
D	K
S'	K'
D'	T

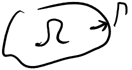
- S'' (and higher) analogously
- Called *layer potentials*:
  - S is called the *single-layer potential*
  - D is called the *double-layer potential*
- (Show pictures using `pytential/examples/layerpot.py`, observe continuity properties.)

How does this actually solve a PDE?

$$S\sigma(x) = \int_\Gamma G(x-y)\sigma(y)\,dS_y$$

$$\Delta G = \delta$$

Solve a (interior Laplace Dirichlet) BVP, $\partial\Omega = \Gamma$



$$\triangle u = 0 \quad \text{in } \Omega, \qquad u|_\Gamma = f|_\Gamma.$$

1. Pick *representation*:

$$u(x) := (S\sigma)(x)$$

$$\Delta (S\sigma)(x) = \int \underbrace{\Delta G(x,y)}_{x\in\Omega \;\to 0} \sigma(y)\,dS_y = 0$$

2. Take (interior) limit onto Γ:

$$u|_\Gamma = S\sigma$$

3. Enforce BC:

$$u|_\Gamma = f$$

4. Solve resulting linear system:

$$\int_\Gamma G(x,y)\sigma(y)\,dS_y \qquad S\sigma = f$$

$$(I - K)$$

(quickly–using the methods we've developed: It is precisely of the form that suits our fast algorithms!)

5. Obtain PDE solution in Ω by evaluating representation

IE BVP Solve: Observations (I)

Observations:

- One can choose representations relatively freely. Only constraints:
  - Can I get to the solution with this representation?
    I.e. is the solution I'm looking for represented?
  - Is the resulting integral equation solvable?

Q: How would we know?

IE BVP Solve: Observations (II)

- Some representations lead to better integral equations than others. The one above is actually terrible (both theoretically and practically). Fix above: Use $u(x) = D\sigma(x)$ instead of $u(x) = S\sigma(x)$.
  Q: How do you tell a good representation from a bad one?
- Need to actually *evaluate* $S\sigma(x)$ or $D\sigma(x)$...
  Q: How?

$\rightarrow$ Need some theory

Outline

Norms

Definition

(Norm) A *norm* $\|\cdot\|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Can create norm from *inner product*: $\|x\| = \sqrt{\langle x, x \rangle}$

Function Spaces

Name some function spaces with their norms.

$C(\Omega)$	f continuous, $\\|f\\|_\infty := \sup_{x\in\Omega} \|f(x)\|$
$C^k(\Omega)$	f k-times continuously differentiable
$C^{0,\alpha}(\Omega)$	$\\|f\\|_\alpha := \\|f\\|_\infty + \sup_{x\neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|^\alpha}$ $(\alpha \in (0,1))$
$C_L(\Omega)$	$\|f(x) - (y)\| \leq L\\|x-y\\|$
$L^p(\Omega)$	$\\|f\\|_p := \sqrt[p]{\int_D \|f(x)\|^p dx} < \infty$
	L^2 special because?

Convergence

Name some ways in which a sequence can 'converge'.

Definition (Convergent sequence)

$x_n \to x :\Leftrightarrow \|x_n - x\| \to 0$ "convergence in norm"

Definition (Cauchy sequence)

For all $\epsilon > 0$ there exists an n for which $\|x_\nu - x_\mu\| \le \epsilon$ for $\mu, \nu \ge n$

(Convergence without known limit!)

Definition (Complete/"Banach" space)

Cauchy $\Rightarrow$ Convergent

Q: Counterexample?

Operators

$A(x+y) = Ax + Ay$

X, Y: Banach spaces, $A : X \to Y$ linear operator

Definition (Operator norm)

$\|A\| := \sup\{\|Ax\| : x \in X, \|x\| = 1\}$

Theorem

$\|A\|$ *bounded* $\Leftrightarrow$ A *continuous*

Other facts?

- The set of bounded linear operators is itself a Banach space: $L(X, Y)$
- $\|Ax\| \leq \|A\|\|x\|$
- $\|BA\| \leq \|B\|\|A\|$

- What does 'linear' mean here?
- Is there a notion of 'continuous at x' for linear operators?

Operators: Examples

Which of these is bounded as an operator on functions on the real line?

- Multiplication by a scalar
- "Left shift"
- Fourier transform $\to L^1/L^2$
- Differentiation $\to C^1(\Omega), \|\cdot\|_\infty$ $\quad C(\Omega)\ \|\cdot\|_\infty$ $\quad \|e^{i\alpha x}\|_\infty = 1$
- Integration $\to$ if Ω is bounded $\quad \|\partial_x e^{i\alpha x}\|_\infty = |\alpha|$
- Integral operators (depends)

Need to know spaces (norms really) to answer that!

Integral Equations: Zoology

Volterra	Fredholm
$\int_a^x k(x,y)f(y)dy = g(x)$	$\int_G k(x,y)f(y)dy = g(x)$

First kind	Second Kind
$\underbrace{\int_G k(x,y)f(y)dy}_{\text{compact}} = g(x)$	$\zeta f(x) + \underbrace{\int_G k(x,y)f(y)dy}_{\text{compact}} = g(x)$

Questions:

- First row: First or second kind?
- Second row: Volterra or Fredholm?
- Matrix (i.e. finite-dimensional) analogs? Volterra – triangular
- What can happen in 2D/3D?
- Factor allowable in front of the identity?
- Why even talk about 'second-kind operators'?
  - Throw a $+\delta(x-y)$ into the kernel, back to looking like first kind. So?
  - Is the identity in $(I+K)$ crucial?

Connections to Complex Variables

Complex analysis is *full* of integral operators:

- Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z-a} f(z)\, dz$$

- Cauchy's differentiation formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{1}{(z-a)^{n+1}} f(z)\, dz$$

Integral Operators: Boundedness (=Continuity)

$G \times G$

Theorem (Continuous kernel $\Rightarrow$ bounded)

$G \subset \mathbb{R}^n$ closed, bounded ("compact"), $K \in C(G^2)$. Let

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

Then

$$\|A\|_\infty = \underbrace{\max_{x \in G} \int_G |K(x,y)|dy}.$$

Show '$\leqslant$'.

$$|A\varphi(x)| \le \int |K(x,y)|\,|\varphi(y)|\,dy$$
$$\le \int |K(x,y)|dy\, \|\varphi\|_\infty$$
$$\le A_\infty \|\varphi\|_\infty$$

Solving Integral Equations

Given

$$(A\phi)(x) := \int_G K(x,y)\varphi(y)dy,$$

are we allowed to ask for a solution of

$$(\mathrm{Id} + A)\varphi = g?$$

Will see three attempts to answer that, in roughly historical order:

- Neumann
- Riesz
- Fredholm