Today

- Neumann
- Riesz
- Fredholm

$$\left(I - \overset{\text{compact}}{A}\right)\varphi = f$$

solvable?

Announcements

- Project proposal submissions?
- Book?

Solving Integral Equations

Given

$$(A\phi)(x) := \int_G K(x,y)\varphi(y)dy,$$

are we allowed to ask for a solution of

$$(\mathrm{Id} + A)\varphi = g?$$

Attempt 1: The Neumann series

Want to solve

$$\varphi - A\varphi = (I - A)\varphi = g.$$

Formally:

$$\varphi = (I - A)^{-1} g.$$

geom. series

$$\frac{1}{1-\alpha} = \sum_{k=0}^{\infty} \alpha^k$$

What does that remind you of?

$|\alpha| < 1$

$\|A\| < 1$

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

Attempt 1: The Neumann series (II)

Theorem

$A : X \to X$ Banach, $\|A\| < 1$ $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$ with $\|(I - A)^{-1}\| \leq 1/(1 - \|A\|)$.

- How does this rely on completeness/Banach-ness?
- There's an iterative procedure hidden in this. (Called *Picard Iteration*. Cf: Picard-Lindelöf theorem.) *Hint:* How would you compute $\sum_k A^k f$?

$\varphi_0 = $ (initial guess)

$\varphi_k = A\varphi_{k-1} + \varphi_{k-1}$

Q: Why does this fall short?

$\|A\| < 1$ is too restrictive

Compact Sets

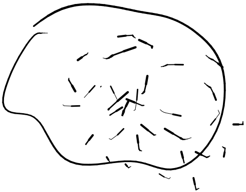
Definition (Precompact/Relatively compact)

$M \subseteq X$ precompact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in X

Definition (Compact/'Sequentially complete')

$M \subseteq X$ compact:$\Leftrightarrow$ all sequences $(x_k) \subset M$ contain a subsequence converging in M

- Precompact $\Rightarrow$ bounded
- Precompact $\Leftrightarrow$ bounded (finite dim. only!)



Compact Sets (II)

Counterexample to 'precompact $\Leftrightarrow$ bounded'? (∞ dim)

$(1, 0, \ldots)$

$(0, 1, \ldots)$

$(0, 0, 1, \ldots)$

Compact Operators

X, Y: Banach spaces

Definition (Compact operator)

$T : X \to Y$ is *compact* $:\Leftrightarrow$ T(bounded set) is precompact.

Theorem

- T, S *compact* $\Rightarrow \alpha T + \beta S$ *compact*
- *One of* T, S *compact* $\Rightarrow S \circ T$ *compact*
- T_n *all compact,* $T_n \to T$ *in operator norm* $\Rightarrow T$ *compact*

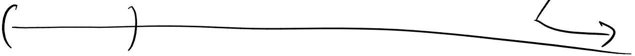
Questions:

- Let $\dim T(X) < \infty$. Is T compact?
- Is the identity operator compact? ($\Leftarrow$) $\dim X < \infty$.

Intuition about Compact Operators

- Compact operator: As finite-dimensional as you're going to get in infinite dimensions.
- Not clear yet–but they are moral (∞-dim) equivalent of a matrix having *low numerical rank*.
- Are compact operators continuous (=bounded)? yes
- What do they do to high-frequency data?
- What do they do to low-frequency data?

compact operators are smoothing



$$\partial_x \left(e^{i\alpha x}\right) = i\alpha$$

Arzelà-Ascoli

Let $G \subset \mathbb{R}^n$ be compact.

Theorem (Arzelà-Ascoli)

$U \subset C(G)$ is precompact iff it is bounded and equicontinuous.

U Equicontinuous means

For all $\varepsilon > 0$ there exists a $\delta > 0$ so that for all $f \in U$

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

f Continuous means:

For all $\varepsilon > 0$ there exists a $\delta > 0$

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

[2cm]

Arzelà-Ascoli: Proof Sketch

$\mathbb{Q}^d$ $\quad$ $0,1$

G

Pick a dense sequence $(x_n) \subset G$.

x_j

G

$(\varphi_{n_i(k)})_k$ $\quad$ $\psi_j = \varphi_{n_j(j)}$

Arzelà-Ascoli: Proof Sketch

bound that ind. of x

$|x_i - x| < \delta$

$$|\psi_j(x) - \psi_k(x)|$$

$$\leq |\psi_j(x) - \psi_j(x_i)| + |\psi_j(x_i) - \psi_k(x_i)|$$

$$+ |\psi_k(x_i) - \psi_k(x)|$$

equi cont.

pw. conv.

Arzelà-Ascoli (II)

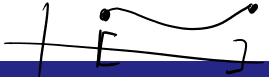
Intuition?

"Uniformly continuous"?

When does *uniform continuity* happen?

(Note: Kress LIE 2nd ed. defines 'uniform equicontinuity' in one go.)

Integral Operators are Compact



Theorem (Continuous kernel $\Rightarrow$ compact [Kress LIE 2nd ed. Thm. 2.20])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. *Then*

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

is compact on $C(G)$.

$\|k\|_\infty \le M$

Pick $U \subseteq C(G)$. k bounded on G

Use A-A. (a statement about compact *sets*) What is there to show?
Pick $U \subset C(G)$. $A(U)$ bounded?

$$|A\varphi(x)| \le \int_G |k(x,y)|\,|\varphi(y)|\,dy \le |G|\,M\|\varphi\|_\infty$$

$A(U)$ equicontinuous?

k uniformly continuous.

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \psi \in A(U)$$

$$|x-y| < \delta \Rightarrow |\psi(x) - \psi(y)| < \varepsilon$$

$$A(\varphi) = \psi$$

$$|\psi(x) - \psi(y)| = |A\varphi(x) - A\varphi(y)| \leq \int \underbrace{|k(x,z) - k(y,z)|}_{\text{use unif cont. to bound that.}} |\varphi(z)| \, dz$$

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n} \qquad (x \neq y)$$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 2nd ed. Thm. 2.22])

K weakly singular. Then

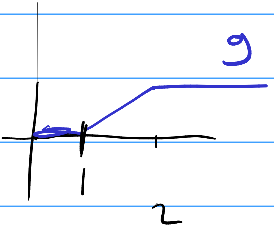
$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

is compact on $C(G)$.

surface area of the unit sphere in n dimensions

$$\int_{B(x,d)} |x-y|^{\alpha-n}\, dy = \omega_n \int_0^d \rho^{\alpha-n} \rho^{n-1}\, d\rho$$

$$= \frac{d^\alpha \omega_n}{\alpha}$$



$$A_n \varphi(x) = \int_G K(x,y)\; g(n|x-y|)\, \varphi(y)\, dy$$

A_n are norm-convergent and each compact $\Rightarrow$ limit is also compact.

Weakly singular: Proof Outline

Outline the proof of 'Weakly singular kernel $\Rightarrow$ compact'.