Today:

- spectral theory
- $\Delta u = 0$ "harmonic" / "potentials"
  - Green's formula
  - "potential theory"
- $\Delta u = 0$ BVP

Announcements:

- HW4
- Projects

$$u(x) = S(\partial_n u) - D(u)$$

Spectral Theory: Terminology

$A : X \to X$ bounded, λ is a . . . value: $Ax = \lambda x \Leftrightarrow (A - \lambda I)x$.

Definition (Eigenvalue)

There exists an element $\phi \in X$, $\phi \neq 0$ with $A\phi = \lambda\phi$.

Definition (Regular value)

The "resolvent" $(\lambda I - A)^{-1}$ exists and is bounded.

Can a value be regular and "eigen" at the same time?

> nope

What's special about ∞-dim here?

> not regular $\Rightarrow$ eigen? $(\lambda I - A)$ → not 1-to-1 $\Rightarrow$ eigenvector
> → not onto ← ? ← A compact → eigenvector

Resolvent Set and Spectrum

Definition (Resolvent set)

$\rho(A) := \{\lambda \text{ is regular}\}$

Definition (Spectrum)

$\sigma(A) := \mathbb{C} \setminus \rho(A)$

Spectral Theory of Compact Operators

$\Delta\Delta u = 0$

$I = A A^{-1}$ compact $\to$ only for finite dim.

Theorem

$A : X \to X$ compact linear operator, X ∞-dim.

Then:

- $0 \in \sigma(A)$ *(show!)* Suppose 0 is a reg. value $\Leftrightarrow$ $(\lambda I - A)^{-1} = A^{-1}$ exists
- $\sigma(A) \setminus \{0\}$ *consists only of eigenvalues* $\Leftarrow$ see above
- $\sigma(A) \setminus \{0\}$ *is at most countable*
- $\sigma(A)$ *has no accumulation point except for 0* $(\Leftarrow)$

$R > 0$ $\{\lambda \in \sigma(A) ; |\lambda| \geq R\}$ finite

Spectral Theory of Compact Operators: Proofs

Show first part.

Show second part.

Spectral Theory of Compact Operators: Implications

Rephrase last two: how many eigenvalues with $|\cdot| \geq R$?

finitely many

Recap: What do compact operators do to high-frequency data?

squish

Don't confuse $I - A$ with A itself!

ok

Outline

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(S'\sigma)(x) := PV\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(D\sigma)(x) := PV \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

$$(D'\sigma)(x) := f.p.\ \hat{n}\cdot\nabla_x \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

Definition (Harmonic function)

$\triangle u = 0$

Where are layer potentials harmonic?

away from Γ

On the double layer again

Is the double layer *actually* weakly singular? Recap:

Definition (Weakly singular kernel)

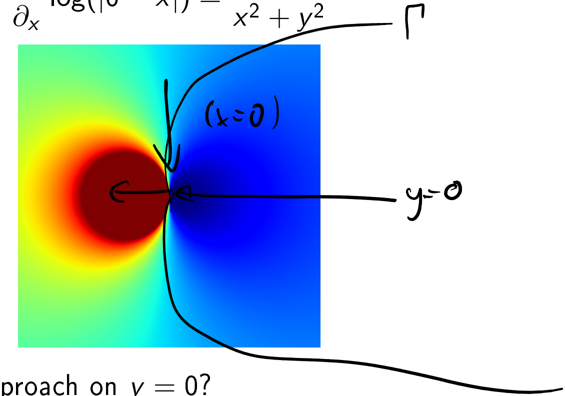
- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \leq C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

$r^{-1+\alpha}$

Actual Singularity in the Double Layer

$$\frac{\partial}{\partial_x} \log(|0 - x|) = \frac{x}{x^2 + y^2}$$



- Singularity with approach on $y = 0$?
- Singularity with approach on $x = 0$?

So life is simultaneously worse and better than discussed.

How about 3D? $(-x/|x|^3)$

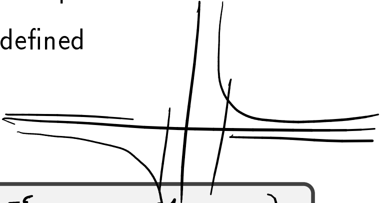
Would like an analytical tool that requires 'less' fanciness.

Cauchy Principal Value

But I don't want to integrate across a singularity! $\rightarrow$ punch it out.

Problem: Make sure that what's left over is well-defined

$$\int_{-1}^{1} \frac{1}{x}\, dx?$$

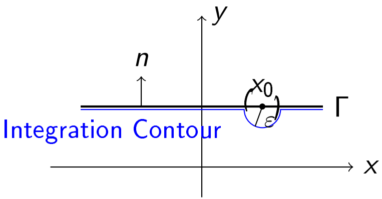


$$\text{P.V.} \int_{-1}^{1} \frac{1}{x}\, dx := \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{-\varepsilon} \frac{1}{x}\, dx + \int_{\varepsilon}^{1} \frac{1}{x}\, dx \right)$$

2ε

no! symmetry matters for boundedness

Principal Value in n dimensions



Again: Symmetry matters!

What about even worse singularities?

Finite part

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(S'\sigma)(x) := \text{PV } \hat{n} \cdot \nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(D\sigma)(x) := \text{PV } \int_\Gamma \hat{n} \cdot \nabla_y G(x-y)\sigma(y)ds_y$$

$$(D'\sigma)(x) := f.p.\ \hat{n} \cdot \nabla_x \int_\Gamma \hat{n} \cdot \nabla_y G(x-y)\sigma(y)ds_y$$

Important for us: Recover 'average' of interior and exterior limit without having to refer to off-surface values.

Green's Theorem

Theorem (Green's Theorem [Kress LIE 2nd ed. Thm 6.3])

$$\int_\Omega u\triangle v + \nabla u \cdot \nabla v = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v)ds$$

$$\int_\Omega \underset{1}{u}\underset{0}{\triangle v} - v\underset{0}{\triangle u} = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v\underbrace{(\hat{n}\cdot\nabla u)}_{0}ds$$

If $\triangle v = 0$ and $u = 1$, then

$$\int_{\partial\Omega} \hat{n}\cdot\nabla v = 0$$

Green's Formula

$\Delta G = \delta$ $\Delta u = 0$

$u = g$

What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green's second identity?

$$\int_\Omega u \triangle v - v \triangle u = \int_{\partial\Omega} u(\hat{n} \cdot \nabla v) - v(\hat{n} \cdot \nabla u) ds$$

$x \in \Omega \Rightarrow v(x)$

$x \notin \Omega \rightarrow 0$

$x \in \partial\Omega \rightarrow \frac{v(x)}{2}$

$\int_{\partial\Omega} G(y-x)(\hat{n} \cdot \nabla v)$

$v(x)$

$$\boxed{\int u \cancel{\Delta v} - v \Delta G(y-x) = S(\hat{n} \cdot \nabla v)(x) - D v(x)}$$

0 $\delta(y-x)$

Theorem (Green's Formula [Kress LIE 2nd ed. Thm 6.5])

If $\triangle u = 0$, *then*

Ω bounded

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in \Omega \\ \frac{u(x)}{2} & x \in \partial\Omega \\ 0 & x \notin \Omega \end{cases}$$

Green's Formula and Cauchy Data

$$S(\partial_n u) - D(u) = \begin{cases} u \\ u/2 \\ 0 \end{cases}$$

Suppose I know 'Cauchy data' $(u|_{\partial\Omega}, \hat{n} \cdot \nabla u|_{\partial\Omega})$ of u. What can I do?

Compute u anywhere

What if Ω is an exterior domain?

... survives but with a small modification.

What if $u = 1$? Do you see any practical uses of this?

$$-D1(x) = \begin{cases} 1 & x \in \Omega \\ 1/2 & x \in \partial\Omega \\ 0 & x \notin \Omega \end{cases}$$

Mean Value Theorem

$\nabla \log(r) = \frac{\vec{x}}{\|x\|^2}$

Theorem (Mean Value Theorem [Kress LIE 2nd ed. Thm 6.7])

If $\Delta u = 0$, $u(x) = \overline{\int}_{B(x,r)} u(y)dy = \overline{\int}_{\partial B(x,r)} u(y)dy$

Define $\overline{\int}$?

$$\overline{\int}_{\Omega} f(x)\,dx = \frac{1}{|\Omega|}\int f(x)\,dx$$

Trace back to Green's Formula (say, in 2D):

$$u(x) = \int(\partial_n u) - D(u) = \log(r)\cancel{\int \partial_n u(y)\,dS_y} - \frac{1}{2\pi r}\int u\,dS_y$$

∂

Maximum Principle

Theorem (Maximum Principle [Kress LIE 2nd ed. 6.9])

If $\triangle u = 0$ on compact set $\bar{\Omega}$:
u attains its maximum on the boundary.

Suppose it were to attain its maximum somewhere inside an open set. . .

What do our *constructed* harmonic functions (layer potentials) do there?