Today

- Ext Dirichlet
- Domains w/ Γ
- Helmholtz

Announcements

- Projects / presentations

Uniqueness of Integral Equation Solutions

Theorem (Nullspaces [Kress LIE 2nd ed. Thm 6.20])

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \mathrm{span}\{1\}$, $N(I/2 + S') = \mathrm{span}\{\psi\}$, *where* $\int \psi \neq 0$.

IE Uniqueness: Proofs (1/3)

Show $N(I/2 - D) = \{0\}$.

IE Uniqueness: Proofs (2/3)

Show $N(I/2 - S') = \{0\}$.

IE Uniqueness: Proofs (3/3)

Show $N(I/2 + D) = \text{span}\{1\}$.

What conditions on the RHS do we get for int. Neumann and ext. Dirichlet?

$\rightarrow$ "Clean" Existence for 3 out of 4.

Ext. Dirichlet

$$u(x) = \tilde{D}\sigma(x)$$

$$\frac{1}{|x|^{n-2}} \int \sigma$$

$$G(x,y) = \hat{n}\cdot\nabla_y G(x,y) + 1$$

$$\tilde{D}\sigma(\vec{x}) = \int_{\partial\Omega} \hat{n}\cdot\nabla_y G(x,y)\,\sigma(y)\,dy + \overbrace{\int_{\partial\Omega} \underbrace{\frac{1}{|x|^{n-2}}}\,\sigma(y)\,dy}^{\text{const}}$$

Patching up Exterior Dirichlet

Problem: $N(I/2 + S') = \{\psi\}\ldots$ do not know ψ. Use different kernel:

$$\hat{n} \cdot \nabla_y G(x, y) \quad \to \quad \hat{n} \cdot \nabla_y G(x, y) + \frac{1}{|x|^{n-2}}$$

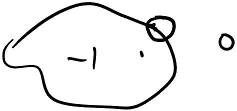
Note: Singularity only at origin! (assumed $\in \Omega$)

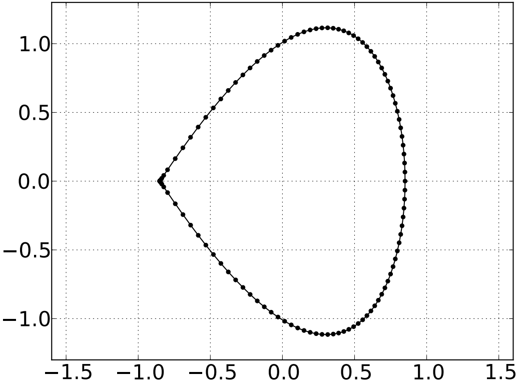
- 2D behavior? 3D behavior?
- Still a solution of the PDE? Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
  - Consider $\partial_n G = O(1/r^{n-1})$.
- $|x|^{n-2}u(x) = ?$ as $|x| \to \infty$?
- Thus $\underline{\int \phi = 0.}$ Contribution of the second term?
- $\phi/2 + D\phi = 0$, i.e. $\phi \in N(I/2 + D) = \text{span}\{1\} \Rightarrow \varphi = 0$
- Existence/uniqueness?

$\to$ Existence for 4 out of 4.

Domains with Corners

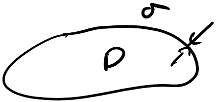
$D1 = -1_\Omega$

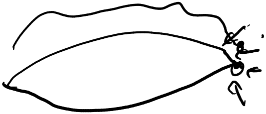




What's the problem? *(Hint: Jump condition for constant density)*

Domains with Corners (II)



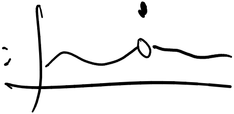


At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x, y) \phi(y) ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on } \pm \text{ side} \rangle}{\pi} \phi$$

$\to$ non-continuous behavior of potential on Γ at x_0

$\mathcal{D}\sigma$:



What space have we been living in?

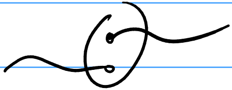
Fixes:

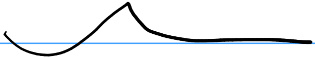
- I + Bounded (Neumann) + Compact (Fredholm)
- Use L^2 theory
  (point behavior "invisible")

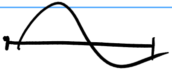
Numerically: Needs consideration, can drive up cost through refinement.

$$L^2(\Omega) = \{ f : \int_\Omega |f|^2 < \infty \} \qquad H^1(\Omega)$$

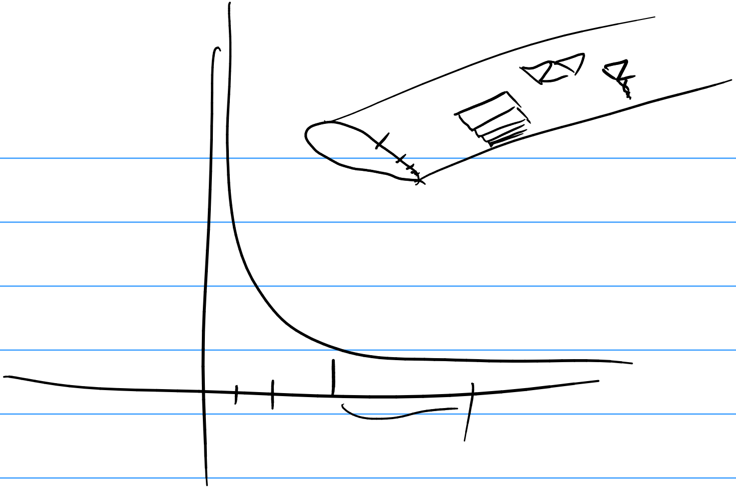
H^1:  ✓

H^1:  ✗

H^1:  ✓



$$S : H^{-1/2} \to H^{1/2}$$



Where does Helmholtz come from?

$u(t,x,y)$

Derive the Helmholtz equation from the wave equation $\partial_t^2 U = c^2 \triangle U,$. Q: What is c?

$$U(x,t) = e^{-i\omega t}\, u(x)$$

$$u(x)\,\partial_t^2\left(e^{-i\omega t}\right) = c^2 e^{-i\omega t} \Delta u$$

$$u(x)(-i\omega)^2 \cancel{e^{-i\omega t}} = c^2 \cancel{e^{-i\omega t}} \Delta u$$

$$-\omega^2 u(x) = c^2 \Delta u$$

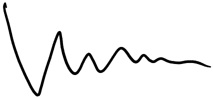
$$u(x) = \underbrace{\frac{c^2}{\omega^2}}_{k^2} \Delta u$$

Helmholtz vs. Yukawa

Helmholtz Equation

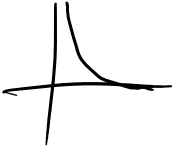
- $\triangle u + k^2 u(x) = 0$
- Indefinite operator
- Oscillatory solution
- Difficult to solve, especially for large k

$$\frac{e^{-ikr}}{r}$$



Yukawa Equation

- $-\triangle u + k^2 u(x) = 0$
- Positive definite operator
- Smooth solutions
- ‘Screened Coulomb’ interaction
- Generally quite simple to solve



Helmholtz vs. Yukawa

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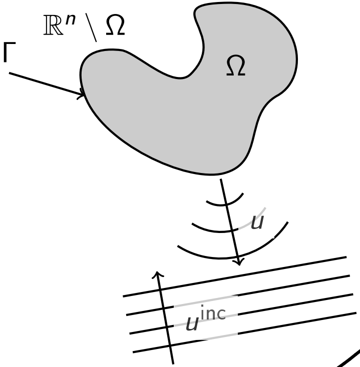
The prototypical Helmholtz BVP: A Scattering Problem



$e^{ik\vec{x}\cdot\vec{n}}$

Ansatz:

$$u^{\text{tot}} = u + u^{\text{inc}}$$

Solve for scattered field u.

Helmholtz: Some Physics

Colton/Kress

Physical quantities:

- Velocity potential: $U(x,t) = u(x)e^{-i\omega t}$ (fix phase by e.g. taking real part)
- Velocity: $v = (1/\rho_0)\nabla U$
- Pressure: $p = -\partial_t U = i\omega u e^{-i\omega t}$
  - Equation of state: $p = f(\rho)$

"sound-hard"
"sound-soft"

What's ρ_0?

density about which we've linearized

What happens to a pressure BC as $\omega \to 0$?

Helmholtz: Boundary Conditions

Interfaces between media: What's continuous?

- Sound-soft: Scatterer "gives"
  - Pressure remains constant in time
  - $u = f \to$ Dirichlet
- Sound-hard: Scatterer "does not give"
  - Pressure varies, same on both sides of interface
  - $\hat{n} \cdot \nabla u = 0 \to$ Neumann
- Impedance: Some pressure translates into motion
  - Scatterer "resists"
  - $\hat{n} \cdot \nabla u + ik\lambda u = 0 \to$ Robin ($\lambda > 0$)
- Sommerfeld radiation condition: allow only outgoing waves (n-dim)

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) u(x) \to 0 \qquad (r \to \infty)$$

Many interesting BCs $\to$ many IEs! :)

$$0 = (\partial_t^2 - \partial_x^2)\, u = \underbrace{(\overset{-i\omega}{\partial_t} - \partial_x)}\,(\overset{-i\omega}{\partial_t} + \partial_x)\, u$$

$$(\partial_t - \partial_x)\, u = 0 \qquad \text{"advection"}$$

$$u(x,t) = \bar{u}(t + x)$$

Unchanged from Laplace

Theorem (Green's Formula [Colton/Kress IAEST Thm 2.1]**)**

If $\triangle u + k^2 u = 0,$ *then*

$$(S(\hat{n} \cdot \nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

$$[Su] = 0$$

$$\lim_{x \to x_0 \pm} (S' u) = \left(S' \mp \frac{1}{2} I \right) (u)(x_0) \quad \Rightarrow \quad [S' u] = -u$$

$$\lim_{x \to x_0 \pm} (Du) = \left(D \pm \frac{1}{2} I \right) (u)(x_0) \quad \Rightarrow \quad [Du] = u$$

$$[D' u] = 0$$

Unchanged from Laplace

Why is singular behavior (esp. jump conditions) unchanged?

$$\frac{e^{ikr}}{r} = \frac{1}{r} + O(1) \quad \text{as } r \to 0$$

Why does Green's formula survive?

$$\int_\Omega u \Delta v - v \Delta u = \int u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u)$$

$$u \to \Delta u + k^2 u = 0$$
$$v \to G$$

Resonances

$$\int_\Omega \Delta u\, v = \int u\, \Delta v$$

$$x^T A x > 0 \qquad \int u \Delta u \geq 0$$

$-\triangle$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.

What does that have to with Helmholtz?

$u=0$

$$-\Delta u = \lambda u \qquad (\lambda > 0)$$

$$\Delta u + k^2 u = 0$$

Why could it cause grief?

int. BVPs: non-uniqueness

Helmholtz: Boundary Value Problems

Find $u \in C(\bar{D})$ with $\triangle u + k^2 = 0$ such that

	Dirichlet	**Neumann**
Int.	$\lim_{x\to\partial D-} u(x) = g$ O unique ($-$resonances)	$\lim_{x\to\partial D-} \hat{n}\cdot\nabla u(x) = g$ O unique ($-$resonances)
Ext.	$\lim_{x\to\partial D+} u(x) = g$ Sommerfeld + unique	$\lim_{x\to\partial D+} \hat{n}\cdot\nabla u(x) = g$ Sommerfeld + unique

with $g \in C(\partial D)$.

Find layer potential representations for each.

int. Dirichlet: $\frac{I}{2} - D$ $\qquad$ $\frac{I}{2} + S'$ int. Neumann

ext. Dirichlet: $\frac{I}{2} + D$

Patching up resonances

Issue: Ext. IE inherits non-uniqueness from 'adjoint' int. BVP

Fix: Tweak representation [Brakhage/Werner '65, ...]
(also called the *CFIE* or *combined field integral equation*)

$$u = D\phi + i\alpha S\phi$$

(α: tuning knob $\to 1$ is fine, $\sim k$ better for large k)