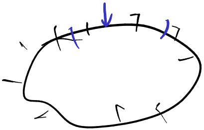


Today

- unstructured mesh demo
- Nyström
- Collective compactness
- $\|A - A_n\| \to 0$
  - Anselone's theorem

Announcements

- Project
- HW4

Nyström Discretizations (I)

Nyström consists of two distinct steps: $\varphi_n(x) = f - \sum \dots$

1. Approximate integral by quadrature: $\int$

$$\varphi_n(x) - \sum_{k=1}^{n} \omega_k K(x, y_k)\varphi_n(y_k) = f(x) \qquad (1)$$

2. Evaluate quadrature'd IE at quadrature nodes, solve discrete system

$$\varphi_j^{(n)} - \sum_{k=1}^{n} \omega_k K(x_j, y_k)\varphi_k^{(n)} = f(x_j) \qquad (2)$$

with $x_j = y_j$ and $\varphi_j^{(n)} = \varphi_n(x_j) = \varphi_n(y_j)$

Is version (1) solvable?

?

tall skinny

Nyström Discretizations (II)

What's special about (2)?

no continuous

Solution density also only known at point values. But: can get approximate continuous density. How?

$$\tilde{\varphi}_n(x) = f(x) - \sum \ldots$$

Assuming the IE comes from a BVP. Do we also only get the BVP solution at discrete points?

$$u = D\varphi \longrightarrow u = \int_\Gamma \partial_n G(x-y)\varphi(y)\,dy$$

$$\hookrightarrow \frac{\varphi}{2} - D\varphi = f \qquad = \sum \partial_n G(x - y_j)\varphi(y_j)\omega_j$$

Nyström Discretizations (III)

Does (1) $\Rightarrow$ (2) hold?

Sure

Does (2) $\Rightarrow$ (1) hold?

If you've chosen φ to be the Nyström interpolant.

Nyström Discretizations (IV)

What good does that do us?

$\|\text{error}\|_\infty$ hard to eval for pt. values

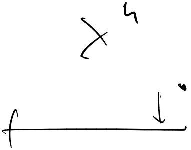
$\varphi - \varphi_n$

we don't have to

Does Nyström work for first-kind IEs?

$A\varphi = f$ no; no Nyström interpolant.

Convergence for Nyström



Increase number of quadrature points n:
Get sequence (A_n)
Want $A_n \to A$ in some sense
What senses of convergence are there for sequences of functions f_n?

- uniform conv. ("norm convergence") $\|f_n - f\| \to 0$
- pointwise $x: f_n(x) \to f(x)$

What senses of convergence are there for sequences of operators A_n?

- norm conv. $\|A - A_n\| \to 0$
- functionwise $\varphi: A_n\varphi \to A\varphi$ ← not strong enough

Convergence for Nyström (II)

$A_n(\varphi \psi_\varepsilon) = 0$

Will we get norm convergence $\|A_n - A\|_\infty \to 0$ for Nyström?

$\psi_\varepsilon = 1$ everywhere except in an ε-nbh of the quad nodes — there it's 0.

$\|A\varphi\psi_\varepsilon - A\varphi\| \to 0$ $(\varepsilon \to 0)$ $\max \left| \int k(x,y)(\psi_\varepsilon - 1)\varphi \right.$

Is functionwise convergence good enough?

$$\|A - A_n\|_\infty \geq \|A\|_\infty$$

Pf:

$$\|A - A_n\| = \sup_{\|\varphi\|=1} \|(A-A_n)\varphi\| \geq \|(A-A_n)\varphi\psi_\varepsilon\|$$

$$= \|A\varphi\psi_\varepsilon - \underbrace{A_n\varphi\psi_\varepsilon}_{0}\| \to \|A\|_\infty$$

Compactness-Based Convergence

X Banach space (think: of functions)

Theorem (Not-quite-norm convergence [Kress LIE 2nd ed. Cor 10.4])

$A_n : X \to X$ bounded linear operators,
functionwise convergent to $A : X \to X$
Then convergence is uniform on compact subsets $U \subset X$, i.e.

$$\sup_{\phi \in U} \|A_n\phi - A\phi\| \to 0 \qquad (n \to \infty)$$

How is this different from norm convergence? $\|A_n\| \leq C$

Only on a compact set

Collective Compactness

Set $\mathcal{A}$ of operators $A : X \to X$

Definition (Collectively compact)

$\mathcal{A}$ is called *collectively compact* if and only if
for $U \subset X$ bounded, $\mathcal{A}(U)$ is relatively compact.

What was relative compactness (=precompactness)?

$\exists$ conv. subseq.

Collective Compactness: Questions (1/2)

Is each operator in the set $\mathcal{A}$ compact?

yes

Is collective compactness the same as "every operator in $\mathcal{A}$ is compact"?

no.

Collective Compactness: Questions (2/2)

When is a sequence collectively compact?

view seq. as set ✓

Is the limit operator of such a sequence compact?

✓

How can we use the two together?

Making use of Collective Compactness

X Banach space, $A_n : X \to X$, (A_n) collectively compact, $A_n \to A$ functionwise.

Corollary (Post-compact convergence [Kress LIE 2nd ed. Cor 10.8])

- $\|(A_n - A)A\| \to 0$
- $\|(A_n - A)A_n\| \to 0$

$(n \to \infty)$

Anselone's Theorem

$$(I - A)\varphi = f$$

Assume:

$(I - A)^{-1}$ exists, with $A : X \to X$ compact, $(A_n) : X \to X$ collectively compact and $A_n \to A$ functionwise.

Theorem (Nyström error estimate [Kress LIE 2nd ed. Thm 10.9])

For sufficiently large n, $(I - A_n)$ is invertible and

$$\|\phi_n - \phi\| \le C(\|(A_n - A)\phi\| + \|f_n - f\|)$$

$$C = \frac{1 + \|(I - A)^{-1}A_n\|}{1 - \|(I - A)^{-1}(A_n - A)A_n\|}$$

$I + (I - A)^{-1}A = ?$

$$1 + \frac{a}{1-a} = \frac{1-a}{1-a} + \frac{a}{1-a} = \frac{1}{1-a}$$