Today

- Nyström
  - ↳ Anselone
  - ↳ collective compactness
  - ↳ $\cancel{\|A_n - A\|} \to 0$

    $A_n \to A$ functionwise

Announcements

- HU 4
- Please make sure project proposals are "green"/submitted

Anselone's Theorem

Assume:
$(I - A)^{-1}$ exists, with $A : X \to X$ compact, $(A_n) : X \to X$ collectively compact and $A_n \to A$ functionwise.

Theorem (Nyström error estimate [Kress LIE 2nd ed. Thm 10.9])

For sufficiently large n, $(I - A_n)$ is invertible and

$$\|\phi_n - \phi\| \le C(\|(A_n - A)\phi\| + \|f_n - f\|)$$

$$C = \frac{1 + \|(I - A)^{-1} A_n\|}{1 - \|(I - A)^{-1}(A_n - A)A_n\|}$$

$A \leftrightarrow A_n$

$I + (I - A)^{-1} A = ?$

$(I - A)^{-1}$

Anselone's Theorem: Proof (I)

Define approximate inverse $B_n = I + (I - A)^{-1} A_n$.

How good of an inverse is it?

$$
\begin{aligned}
\mathrm{Id} &\approx^{?} B_n(I - A_n) \\
&= (I + (I - A)^{-1} A_n)(I - A_n) \\
&= [I + (I - A)^{-1} A_n] - [A_n + (I - A)^{-1} A_n A_n] \\
&= [I + (I - A)^{-1} A_n] - [(I - A)^{-1}(I - A) A_n + (I - A)^{-1} A_n A_n] \\
&= [I + (I - A)^{-1} A_n] - [(I - A)^{-1} I A_n - (I - A)^{-1} A A_n + (I - A)^{-1} A_n A_n] \\
&= I + (I - A)^{-1} \mathrm{A} A_n - (I - A)^{-1} A_n A_n \\
&= I + \underbrace{(I - A)^{-1}(A - A_n) A_n}_{-S_n} = I - S_n
\end{aligned}
$$

Anselone's Theorem: Proof (II)

$\|A - A_n\| \not\to 0$

Want $S_n \to 0$ somehow. Prior result gives us $\|(A - A_n)A_n\| \to 0$.

$\|S_n\| \to 0$

exists!

$$\|(I - S_n)^{-1}\| \le \frac{1}{1 - \|S_n\|} \qquad \frac{1}{1-0}$$

If $\|S_n\| < 1$. (There must exist an n so that that's the case.)

$$B_n \underbrace{(I - A_n)}_{\text{invertible!}} = \underbrace{I - S_n}_{\text{invertible}}$$

$$(I - A_n)^{-1} = \underbrace{(I - S_n)^{-1} B_n}$$

$$(I - A_n)\varphi_n = f_n$$

Anselone's Theorem: Proof (III)

Let φ be the solution of $(I-A)\varphi = f$ and φ_n the sol. of $(I-A_n)\varphi_n = f_n$

$$(I-A_n)(\varphi_n-\varphi) = f_n - (I-A_n)\varphi$$

$$= f_n - (I-A)\varphi + (A-A_n)\varphi$$

$$= f_n - f + (A-A_n)\varphi_n$$

$$\|\varphi_n-\varphi\| \le \|(I-A_n)^{-1}\| \left(\|f_n-f\| + \|(A_n-A)\varphi\|\right)$$

$$\le \frac{\|B_n\|}{1-\|S_n\|}$$

Anselone: A Question

Nyström: *specific to I + compact.* Why?

Second kind allows solving for the density.

Nyström: Collective Compactness

$$\int_0^1 f(x) = \sum_{i=1}^{n} w_i f(\xi_i) \qquad \sum |w_i|$$

We *assumed* collective compactness. Do we have that? Assume

$$\sum |\text{quad. weights for } n \text{ points}| \leq C \qquad (\text{independent of } n) \qquad (3)$$

- bounded ✓
- equicontinuous (same δ indep. of n, x)

$$\|A_n \varphi\| = \left\| \sum_i w_i\, k(x, y_i)\, \varphi(y_i) \right\|$$

$$|A_n \varphi(x_1) - A_n \varphi(x_2)|$$

$$= \left| \sum_{i=1}^{n} w_i \big(k(x_1, y_i) - k(x_2, y_i)\big)\, \varphi(y_i) \right|$$

$$\leq \sum |w_i| \, \underbrace{|K(x_1, y_i) - k(x_2, y_i)|}_{\text{cont. kernel} \Rightarrow \text{bounded on compact set} \Rightarrow \text{bounded}} \, \|\varphi\|_\infty$$

$$(I - A)^{-1}$$

Nyström: Collective Compactness

$\Rightarrow$ Collective compactness

Also assumed functionwise uniform convergence, i.e. $\|A_n\phi - A\phi\| \to 0$ for each ϕ.

Quad. rule conv. gives x-pw. conv.
Analogous pf. to equicont. gives us x-unif. conv.

Projection Method

$$(I-B)\varphi = f \qquad A\varphi = f$$

X Banach space, $U \subset X$ nontrivial subspace, $A : X \to Y$ injective, $X_n \subset X$, $Y_n \subset Y$, $\dim X_n = n$, $\dim Y_n = n$, $P_n : ? \to ?$

$P_n : Y \to Y_n$

- P is a projection $\Leftrightarrow P|_U = \mathrm{Id} \Leftrightarrow P^2 = P$
- $\|P\| \geq 1$
- Orthogonal projectors: $\|P\| = 1$
- Interpolators ("collocation projection"): Also projections
- Projection method: $P_n A \phi_n = P_n f$ (#)

Define convergence:

there exists an n_0 so that for $n \geq n_0$
- for $f \in A(X)$, (#) has a unique solution
- $\|\varphi_n - \varphi\| \to 0$

Assumptions on the Approximation Spaces

What's needed of X_n so that it can even approximate the solution?

Denseness

$$\inf_{\psi \in X_n} \|\psi - \varphi\| \to 0 \quad (n \to \infty)$$

Error Estimates for Projection

$\varphi_n = (P_n A)^{-1} P_n A \varphi$

$P_n A \varphi_n = P_n f = P_n A \varphi$

X Banach space, $A : X \to X$ injective, $P_n : Y \to Y_n$

Theorem (Céa's Lemma [Kress LIE 2nd ed. Thm 13.6])

Convergence of the projection method $\Leftrightarrow$
There exist n_0 *and* M *such that for* $n \geq n_0$

1. $P_n A : X_n \to Y_n$ *are invertible,*
2. $\underbrace{\|(P_n A)^{-1} P_n A\|} \leq M$. *(Uniform Boundedness, Stability)*

In this case,

$$\|\phi_n - \phi\| \leq (1 + M) \inf_{\psi \in X_n} \|\phi - \psi\|$$

$\int S \varphi + \int_{\partial_{nn}} k(x,y) \varphi(y)\, dy$

Céa's Lemma: Proof

Proof?

Core message of the theorem?