Ann:

- project proposals
  - pres: May 7, 9
  - due: May 10
- HW4

Goals:

- spectral review
- harmonic functions $\Delta u = 0$
  - ↳ one-liner proofs

Review:

▷ $(I - A)\varphi = g$ solvability
- ↳ Riesz
- ↳ Fredholm

$$(I-A)(X) = N(I-A^*)^{\perp}$$

$$(I-A)^* = I^* - A^* = I - A^*$$

$$(Ix, y) = (x, Iy)$$

▷ spectral

Spectral Theory: Terminology

$A : X \to X$ bounded, λ is a ... value:

Definition (Eigenvalue)

There exists an element $\phi \in X$, $\phi \neq 0$ with $A\phi = \lambda\phi$. $\iff \varphi \in N(\lambda I - A)$

Definition (Regular value)

The "resolvent" $(\lambda I - A)^{-1}$ exists and is bounded.

Can a value be regular and "eigen" at the same time?

no

What's special about ∞-dim here?

$\lambda \in \mathbb{C}$ neither eigen nor regular

Resolvent Set and Spectrum

Definition (Resolvent set)

$\rho(A) := \{\lambda \text{ is regular}\}$ $\subseteq \mathbb{C}$ $\iff$ $\{\lambda : (\lambda I - A)^{-1} \text{ exists}\}$

Definition (Spectrum)

$\sigma(A) := \mathbb{C} \setminus \rho(A)$ $\subseteq \mathbb{C}$ $\iff$ $\{\lambda : (\lambda I - A)^{-1} \nexists \text{ or unbounded}\}$

Spectral Theory of Compact Operators

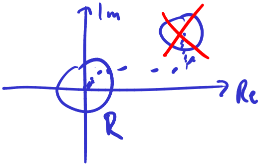
$$\partial_x : \quad \partial_x e^{i\alpha x} = \underbrace{(i\alpha)}_{\lambda} e^{i\alpha x} \quad \text{eigenfunc for all } \alpha \in \mathbb{C}$$

Theorem

$A : X \to X$ compact linear operator, X ∞-dim.

Then:

- $0 \in \sigma(A)$
- $\sigma(A) \setminus \{0\}$ *consists only of eigenvalues*
- $\sigma(A) \setminus \{0\}$ *is at most countable*
- $\sigma(A)$ *has no accumulation point except for 0*



Spectral Theory of Compact Operators: Proofs

eigenv. $Ax = \lambda x$ $(x \neq 0)$

Show the first part. $0 \in \sigma(A)$

$\Leftrightarrow$ 0 is not a regular value. $\Leftrightarrow$ A^{-1} does not exist or is not bounded.
Suppose A^{-1} existed and is bounded. $I = AA^{-1}$ compact $\Leftrightarrow$ dim $< \infty$.
compact bounded

Show second part.

Let $\lambda \in \sigma(A)$ $\Leftrightarrow$ $(\lambda I - A)^{-1}$ does not exist and is not bounded

$\lambda I - A$ is not bijective

$\exists \varphi: (\lambda \varphi - A\varphi) = 0$
$\Leftrightarrow$ φ is eigen.

not injective
$\Leftrightarrow$ has nullspace

not surjective
Riez

Toslow: A injective $\Leftrightarrow$ $N(A) = \{0\}$ (A linear)

"$\Rightarrow$" $\forall x, y: Ax = Ay \Rightarrow x = y$

($\Leftrightarrow$ $\forall x \neq y: Ax \neq Ay$)

$A(x-y) = 0 \Rightarrow x - y = 0$

$\forall z \quad Az = 0 \Rightarrow z = 0$

$\Rightarrow \quad N(A) = \{0\}$

Spectral Theory of Compact Operators: Implications

Rephrase last two: how many eigenvalues with $|\cdot| \geq R$?

finitely many

Recap: What do compact operators do to high-frequency data?

squish

Don't confuse $I - A$ with A itself!

Outline

Outline

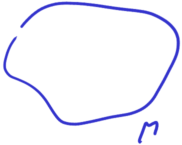
Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(S'\sigma)(x) := PV\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(D\sigma)(x) := PV \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

$$(D'\sigma)(x) := f.p.\ \hat{n}\cdot\nabla_x \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$



$\Delta G = \delta$

Definition (Harmonic function)

$\triangle u = 0$ (at x)

Where are layer potentials harmonic? $G = \frac{1}{2\pi}\log$ / $\frac{1}{4\pi}\frac{1}{r}$

away from Γ

$$(x \notin \Gamma)$$

$$\Delta_x \int_\Gamma G(x-y)\,\sigma(y)\,ds_y$$

$$= \int \underbrace{\Delta_x G(x-y)}_{\delta(x-y)}\,\sigma(y)\,ds_y$$

$$= \int \delta(x-y)\,\sigma(y)\,dS_y = 0$$

On the double layer again

(surface)

Is the double layer *actually* weakly singular? Recap:

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x,y)| \le C|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \ne y)$$

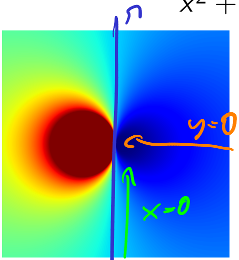
weakly singular ⇒ A compact

$\frac{1}{r^{0.999}}$ is OK (2D)

$\frac{1}{r}$ is not

Actual Singularity in the Double Layer (2D)

$$\frac{\partial}{\partial_x} \log(|0 - x|) = \frac{x}{x^2 + y^2}$$



$$\frac{x}{x^2+0} = \frac{1}{x}$$

x=0

- Singularity with approach on $y = 0$?
- Singularity with approach on $x = 0$?

2D only: the double layer has a removable singularity

Cauchy Principal Value

But I don't want to integrate across a singularity! $\rightarrow$ punch it out.

Problem: Make sure that what's left over is well-defined

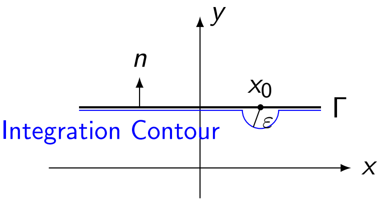
$$\int_{-1}^{1} \frac{1}{x} dx?$$

$$PV \int_{-1}^{1} \frac{1}{x} dx = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{1}{x} dx + \int_{\varepsilon}^{1} dx$$

defined via cancellation of blow-ups.

$$PV \int_{-1}^{1} \frac{1}{x} dx = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{1}{x} dx + \int_{2\varepsilon}^{1} dx$$

Principal Value in n dimensions



Again: Symmetry matters!

has to be a circle.

What about even worse singularities?

f.p.

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(S'\sigma)(x) := \text{PV}\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\sigma(y)ds_y$$

$$(D\sigma)(x) := \text{PV} \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\sigma(y)ds_y$$

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Important for us: Recover 'average' of interior and exterior limit without having to refer to off-surface values.

(to be shown)

Outline

Green's Theorem

Ω bounded

Theorem (Green's Theorem [Kress LIE 2nd ed. Thm 6.3])

$$\int_\Omega u\triangle v + \nabla u \cdot \nabla v = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v)ds$$

$$\int_\Omega u\triangle v - v\triangle u = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u)ds$$

If $\triangle v = 0$ and $u = 1$, then

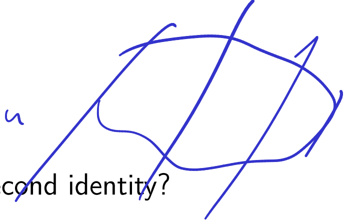
div grad v=0

$$\int_{\partial\Omega} \hat{n}\cdot\nabla v = ?$$

$$\int_\Omega u\triangle v - v\triangle u = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u)ds$$

=0 0 1 0

Green's Formula



What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green's second identity?

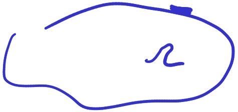
$$\int_{\Omega} u\triangle v - v\triangle u = \int_{\partial\Omega} u(\hat{n}\cdot\nabla v) - v(\hat{n}\cdot\nabla u)ds$$

Can you write that more briefly?

$$-\int_{\Omega} v(y)\,\delta(x-y)\,dy = \int_{\partial\Omega} G(x-y)\,\partial_n v - \int_{\partial\Omega} \partial_n G\, v$$

$$\begin{cases} v(x) & x\in\Omega \\ 0 & x\notin\Omega \end{cases} = S(\partial_n v) - D(v)$$

Green's Formula (Full Version)



Ω bounded

Theorem (Green's Formula [Kress LIE 2nd ed. Thm 6.5])

If $\triangle u = 0$, *then*

$$(S(\hat{n}\cdot\nabla u) - Du)(x) = \begin{cases} u(x) & x \in \Omega, \\ \frac{u(x)}{2} & x \in \partial\Omega, \\ 0 & x \notin \Omega. \end{cases}$$

Green's Formula and Cauchy Data

Suppose I know 'Cauchy data' $(u|_{\partial\Omega}, \hat{n} \cdot \nabla u|_{\partial\Omega})$ of u. What can I do?

What if Ω is an exterior domain?

What if $u = 1$? Do you see any practical uses of this?