April 25, 2024

Announcements

Goals

Review

Helmholtz

"sound hard" $\rightsquigarrow$ Neumann $\partial_n u = 0$

"sound soft" $\rightsquigarrow$ Dirichlet $u = g$

$\nabla u =$ velocity

$-\partial_t u =$ pressure

Helmholtz: Boundary Conditions

Interfaces between media: What's continuous?

normal velocity, pressure

- Sound-soft: Scatterer "gives"
  - Pressure remains constant in time
  - $u = f \rightarrow$ Dirichlet
- Sound-hard: Scatterer "does not give"
  - Pressure varies, same on both sides of interface
  - $\hat{n} \cdot \nabla u = 0 \rightarrow$ Neumann
- Impedance: Some pressure translates into motion
  - Scatterer "resists"
  - $\hat{n} \cdot \nabla u + ik\lambda u = 0 \rightarrow$ Robin ($\lambda > 0$)
- Sommerfeld radiation condition: allow only outgoing waves (n-dim)

$$r^{\frac{n-1}{2}}\left(\frac{\partial}{\partial r} - ik\right)u(x) \rightarrow 0 \qquad (r \rightarrow \infty)$$

$u_{tt} = u_{xx}$

adv. to the right towards ∞

$(\partial_r + \partial_t)u = 0$

$\partial_t u + \partial_x u = 0$

Many interesting BCs $\rightarrow$ many IEs! :)

Unchanged from Laplace

Theorem (Green's Formula [Colton/Kress IAEST Thm 2.1])

If $\triangle u + k^2 u = 0$, *then*

$$(S(\hat{n}\cdot\nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

$$[Su] = 0$$

$$\lim_{x\to x_0\pm}(S'u) = \left(S' \mp \frac{1}{2}I\right)(u)(x_0) \quad \Rightarrow \quad [S'u] = -u$$

$$\lim_{x\to x_0\pm}(Du) = \left(D \pm \frac{1}{2}I\right)(u)(x_0) \quad \Rightarrow \quad [Du] = u$$

$$[D'u] = 0$$

Unchanged from Laplace

Why is singular behavior (esp. jump conditions) unchanged?

$$\frac{1}{r} \to \frac{e^{ikr}}{r} = \frac{1}{r} + \frac{ikr}{r\,1!} + \frac{(ikr)^2}{r\,2!} + \cdots$$

Why does Green's formula survive?

Laplace

$\Delta v + k^2 v = 0$

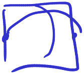
$$\int u\Delta v - v\Delta u = \int u\,\partial_n v - v\,\partial_n u$$

$u = G_k$

$U(x,t) = u(x)e^{-i\omega t}$

if $\omega \in \mathbb{R}$, oscillates. if $\omega \in \mathbb{C}$

Resonances



$-\Delta u = \lambda u$ not $-\Delta u = 0$

$\left.\begin{array}{l} u = 0 \\ \text{or } \partial_n u = 0 \end{array}\right\}$ on $\partial\Omega$

$-\triangle$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.
What does that have to with Helmholtz?

> 0

$-\Delta u = \lambda u \iff \Delta u + \lambda u = 0$

$\Delta u + k^2 u = 0$

resonance!

Why could it cause grief?

$\Delta u + k^2 u = 0$

$u = g$ on $\partial\Omega$

countably many Helmholtz problems are non-unique.

Helmholtz: Boundary Value Problems

Laplace:

	D	N
Int	D	S
Ext	D	S

Find $u \in C(\bar{D})$ with $\triangle u + k^2 = 0$ such that

	Dirichlet	**Neumann**
Int.	$\lim_{x \to \partial D-} u(x) = g$ unique ($-$resonances)	$\lim_{x \to \partial D-} \hat{n} \cdot \nabla u(x) = g$ unique ($-$resonances)
Ext.	$\lim_{x \to \partial D+} u(x) = g$ Sommerfeld unique	$\lim_{x \to \partial D+} \hat{n} \cdot \nabla u(x) = g$ Sommerfeld unique

with $g \in C(\partial D)$.

Find layer potential representations for each.

Repr.

	D	N
Int	D	S
Ext.	D	S

✓ have spurious resonances because they're adjoints of int. operators w/ nullspace

Patching up spurious resonances inherited from adjoint

Issue: Exte IE inherits non-uniqueness from 'adjoint' int. BVP.

Ext. Dirichlet.

$$u(x) := D\varphi - i\alpha S\varphi$$

Combined Field IE "CFIE". $\alpha = 1$.

$\alpha \sim k$ better for high freq.

Patching up resonances: CFIE (1/3)

$\alpha = -1$

$u(x) := D\varphi - iS\varphi$

IE: $\frac{\varphi}{2} + D\varphi - iS\varphi = g$

Suppose $\frac{\varphi}{2} + D\varphi - iS\varphi = 0$. To show: $\varphi = 0$

$\lim_{\partial\Omega^c} u(x) = \qquad = 0 \qquad \Rightarrow u \equiv 0$ on ext. vol.

Patching up resonances: CFIE (2/3)

$u = D\varphi - iS\varphi$

$0 = (\partial_n u)^- = [\partial_n u] = [\quad D'\varphi - iS'\varphi)] = i\varphi$

$0 = u^- = [u] = [D\varphi - iS\varphi) = \varphi$

Patching up resonances: CFIE (3/3)

$z \in \mathbb{C}$
$z\bar{z} = |z|^2$

$-i(\partial_n u)^- = u^-$

Green's first thm: $\int u \Delta v + \nabla u \cdot \nabla v = \int_\partial u \partial_n v$

$v = \bar{u}$

$$\underbrace{\int -k^2|u|^2 + |\nabla u|^2}_{\in \mathbb{R}} = \int_\Omega u \Delta \bar{u} + |\nabla u|^2 = \int_{\partial\Omega} u^- \partial_n(\bar{u})^-$$

$$= -i \int_{\partial\Omega} |u^-|^2$$

$\int_{\partial\Omega} |u^-|^2 = 0 \Rightarrow u^- = 0.$

$u(x) := D\varphi - iS\varphi$

$0 = u^+ - u^- = [u] = \varphi = 0.$

Helmholtz Uniqueness

D's: second kind

Uniqueness for remaining IEs similar:

ext. Neumann $\quad u(x) := S\varphi - i D\varphi$

$$(\partial_n u)^{\pm} = \pm\frac{\varphi}{2} + S'\varphi - i D'\varphi$$

$$\varphi = S\psi$$

Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

Rank and Smoothness

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems
- Laplace
- Helmholtz
- Calderón identities

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

D' is Self-Adjoint

Show that D' is self-adjoint. [Kress LIE 3rd ed. Sec 7.6]

To show: $(D'\varphi, \psi) = (\varphi, D'\psi)$

$u := D\varphi \quad v := D\psi$

Green's second thm: (int. and ext.!)

$$\int_{\partial\Omega} (\partial_n u) v = \int u (\partial_n v)$$

$$(D'\varphi, \psi) = (\partial_n u, [v])_{\partial\Omega} = (\partial_n u, v^+ - v^-)$$

$$= (u^+, (\partial_n v)^+) - (u^-, (\partial_n v)^-)$$

$$= ([u], \partial_n v) = (\varphi, D')$$

Towards Calderón

Show that $(S\varphi, D'\psi) = ((S' + I/2)\varphi, (D - I/2)\psi)$. [Kress LIE 3rd ed. Sec 7.6]

$(\varphi, SD'\psi)$?

Calderón Identities: Summary

$S\tilde{T}$

- $SD' = D^2 - I/4$
- $D'S = S'^2 - I/4$

Also valid for Laplace (jump relation same after all!)

Why do we care?

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Boundary Value Problems

Back from Infinity: Discretization
- Fundamentals: Meshes, Functions, and Approximation
- Integral Equation Discretizations
- Integral Equation Discretizations: Nyström
- Integral Equation Discretizations: Projection

Computing Integrals: Approaches to Quadrature

Going General: More PDEs