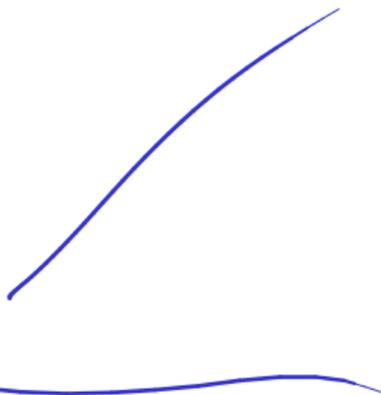


Ann:



Goals:

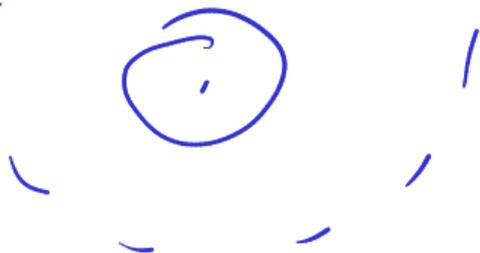
- poles
- comparing ranks
- more expansion types
- "proxy trick"
- fun. alys

Review:

local expn:



Sources



$$\text{err} \sim \left( \frac{d(c, \text{f.t.})}{d(c, \text{c.s.})} \right)^{k+1}$$

multiple poles  
exp.

$$\text{err} \sim \left( \frac{d(c, ?)}{d(c, ?)} \right)^{k+1}$$

## Taylor on Potentials, Again

Stare at that Taylor formula again.

local:  $\psi(\vec{x}-\vec{y}) = \sum_{|\vec{p}| \leq k} \underbrace{\frac{D^{\vec{p}} \psi(\vec{x}-\vec{y})}{\vec{p}!}}_{\text{dep. on sr.}} \underbrace{(\vec{x}-\vec{c})^{\vec{p}}}_{\text{dep. on tgt.}}$

unpole:  $\psi(\vec{x}-\vec{y}) = \sum_{|\vec{p}| \leq k} \underbrace{\frac{D^{\vec{p}} \psi(\vec{x}-\vec{y})}{\vec{p}!}}_{\text{dep. on tgt.}} \underbrace{(\vec{y}-\vec{c})^{\vec{p}}}_{\text{dep. on src.}}$

*basis* (over the fraction in the unpole equation)

*coeff.* (over the term  $(\vec{y}-\vec{c})^{\vec{p}}$  in the unpole equation)

# Multipole Expansions (I)

At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

**First Q:** When does this expansion converge?

$$\sim R^{-|\vec{p}|}$$

$$|p| \gg 1$$

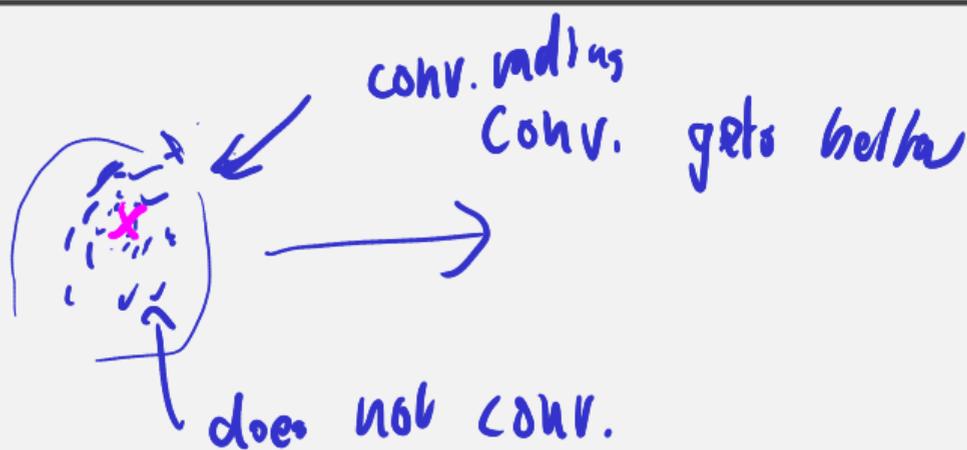
$$\left| \frac{D^{\vec{p}} \psi(\vec{x}-\vec{y})}{\vec{p}!} (\vec{y}-\vec{c})^{\vec{p}} \right| \leq C_{\vec{p}} \left| \frac{\|\vec{y}-\vec{c}\|_2}{\|\vec{x}-\vec{c}\|_2} \right|^{|\vec{p}|}$$

mpole:  $\text{err} \sim \left( \frac{d(\vec{c}, \text{furtherst src})}{d(\vec{c}, \text{closest tgt})} \right)^{|\vec{p}|}$

local:  $\text{err} \sim \left( \frac{d(\vec{c}, \text{furtherst tgt})}{d(\vec{c}, \text{closest src})} \right)^{|\vec{p}|}$

## Multipole Expansions (II)

The abstract idea of a *multipole expansion* is that:



## Multipole Expansions (III)

If our particle distribution is like in the figure: is a multipole expansion is a computationally useful thing?

Set

- ▶  $S = \#$ sources,
- ▶  $T = \#$ targets,
- ▶  $K = \#$ terms in expansion.

Form mpole:  $O(KS)$   
eval mpole:  $O(KT)$       if  $K = O(1)$

**Demo:** Multipole/local expansions

$$\begin{aligned} \lim_{h \rightarrow 0} & \frac{1}{h} \psi(\vec{x} - \frac{h}{2} \vec{e}_x) + \frac{1}{h} (-1) \psi(\vec{x} + \frac{h}{2} \vec{e}_x) \\ & = \pm \partial_x \psi(\vec{x}) \end{aligned}$$

# Outline

Introduction

Dense Matrices and Computation

Tools for Low-Rank Linear Algebra

## Rank and Smoothness

Local Expansions

Multipole Expansions

**Rank Estimates**

Proxy Expansions

Near and Far: Separating out High-Rank Interactions

Outlook: Building a Fast PDE Solver

Going Infinite: Integral Operators and Functional Analysis

Singular Integrals and Potential Theory

Boundary Value Problems

Back from Infinity: Discretization

Computing Integrals: Approaches to Quadrature

Going General: More PDEs

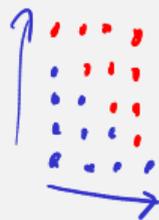
## Taylor on Potentials: Low Rank?

Connect this to the numerical rank observations:

# terms in Taylor:

$$|\vec{p}| \leq k$$

2D: 
$$\frac{(k+1)(k+2)}{2} = O(k^2)$$



3D: 
$$\frac{(k+1)(k+2)(k+3)}{2 \cdot 3} = O(k^3)$$

## On Rank Estimates

So how many terms do we need for a given precision  $\epsilon$ ?

$$\rightarrow \epsilon \approx \left( \frac{d(c, f^*)}{d(c, c_{\text{opt}})} \right)^{k+1} = \rho^{k+1}$$

$$2D: \quad K = k^2$$

$\uparrow$  # terms       $\uparrow$  order of the expr.

$$\log \epsilon \approx (\sqrt{K} + 1) \log \rho$$

$$\sqrt{k+1} \approx \frac{\log \epsilon}{\log \rho}$$

$$\text{\# terms} \xrightarrow{\text{"rank"}} K = \left( \frac{\log \epsilon}{\log \rho} - 1 \right)^2$$

Demo: Checking rank estimates

## Estimated vs Actual Rank

Our rank estimate was off by a power of  $\log \epsilon$ . What gives?

