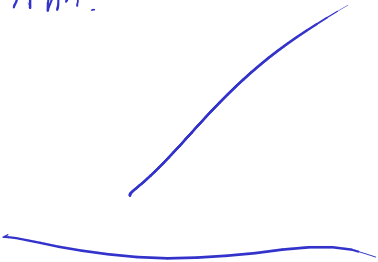
Ann:



Goals:

- mpoles
- comparing ranks
- more expansion types
- "proxy trick"
- fast. algs

Review:

local expn:


$$\text{err} \sim \left( \frac{d(c, \text{f.t.})}{d(c, \text{c.s})} \right)^{k+1}$$

multipole exph.

$$\text{err} \sim \left( \frac{d(c, ?)}{d(c, ?)} \right)^{k+1}$$

# Taylor on Potentials, Again

Stare at that Taylor formula again.

local:

$$\psi(\vec{x}-\vec{y}) = \sum_{|\vec{p}|\le k} \underbrace{\frac{D^{\vec{p}}\psi(\vec{x}-\vec{y})\big|_{\vec{x}=\vec{c}}}{\vec{p}!}}_{\text{dep. on src.}} \underbrace{(\vec{x}-\vec{c})^{\vec{p}}}_{\text{dep. on tgt.}}$$

mpole:

$$\psi(\vec{x}-\vec{y}) = \sum_{|\vec{p}|\le k} \overbrace{\underbrace{\frac{D^{\vec{p}}\psi(\vec{x}-\vec{y})\big|_{\vec{y}=\vec{c}}}{\vec{p}!}}_{\text{dep. on tgt.}}}^{\text{basis}} \overbrace{\underbrace{(\vec{y}-\vec{c})^{\vec{p}}}_{\text{dep. on src.}}}^{\text{coeff.}}$$

# Multipole Expansions (I)

At first sight, it doesn't look like much happened, but mathematically/geometrically, this is a very different animal.

First Q: When does this expansion converge?

$" \mathbb{Q}^{-|\vec{p}|}"$

$|p| \geq 1$

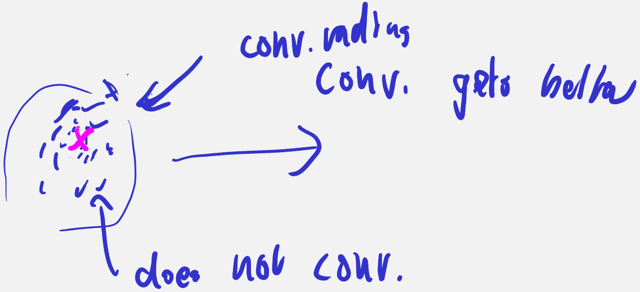
$$\left| \frac{D^{\vec{p}} \psi(\vec{x}-\vec{y}) \big|_{\vec{y}=\vec{c}}}{\vec{p}!} (\vec{y}-\vec{c})^{\vec{p}} \right| \leq C_{\vec{p}} \left| \frac{\|\vec{y}-c\|_2}{\|\vec{x}-\vec{c}\|_2} \right|^{|\vec{p}|}$$

mpole: $\quad err \sim \left( \frac{d(\vec{c}, \text{furthest src})}{d(\vec{c}, \text{closest tgt})} \right)^{k+1}$

local: $\quad err \sim \left( \frac{d(c, \text{furthest tgt})}{d(c, \text{closest src})} \right)^{k+1}$

# Multipole Expansions (II)

The abstract idea of a *multipole expansion* is that:



# Multipole Expansions (III)

If our particle distribution is like in the figure: is a multipole expansion is a computationally useful thing?
Set

- $S$ = #sources,
- $T$ = #targets,
- $K$ = #terms in expansion.

Form mpole: $O(KS)$
eval mpole: $O(KT)$

if $K = O(1)$

**Demo:** Multipole/local expansions

$$\lim_{h\to 0} \frac{1}{h}\psi\left(\vec{x}-\frac{h}{2}\vec{e}_x\right)+\frac{1}{h}(-1)\,\psi\left(x+\frac{h}{2}\vec{e}_x\right)$$

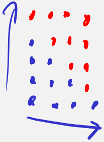
$$= \frac{?}{}\ \partial_x \psi(\vec{x})$$

# Outline

# Taylor on Potentials: Low Rank?

Connect this to the numerical rank observations:

\# terms in Taylor: $|\vec{p}| \le k$



2D: $\frac{(k+1)(k+2)}{2} = O(k^2)$

3D: $\frac{(k+1)(k+2)(k+3)}{2 \cdot 3} = O(k^3)$

# On Rank Estimates

So how many terms do we need for a given precision $\varepsilon$?

$$\to \quad \varepsilon \approx \left(\frac{d(c, ft)}{d(c, cs)}\right)^{k+1} = \rho^{k+1}$$

2D: $K = k^2$

$\uparrow$ # terms $\quad$ $\uparrow$ order of the expn.

$$\log \varepsilon \approx \left(\sqrt{K} + 1\right) \log \rho$$

$$\sqrt{K} + 1 \approx \frac{\log \varepsilon}{\log \rho}$$

# terms "rank" $\to$ $K = \left(\frac{\log \varepsilon}{\log \rho} - 1\right)^2$

**Demo:** Checking rank estimates

# Estimated vs Actual Rank

Our rank estimate was off by a power of $\log \varepsilon$. What gives?