Ann:

- Project prop.
- HW4

Goals:

- Sets of functions $(A \cdot A)$ compact?
- Int. op. compact
- Second-kind $\Rightarrow \exists$ sol.

Review:

$(I-A)\varphi = \varphi - A\varphi = 0 \Leftarrow ② \quad ① \rightarrow A\varphi = 0$

compact

$g(x) = \int_G k(x,y) f(x) dx$ — FrOA.

$g(x) = \int_{G(x)} \dots$ — Volt.

$U \subseteq X$ precompact $:\Leftrightarrow$

$\forall (\varphi_n)_n \subseteq U \Rightarrow \exists n(k)$ so that $(\varphi_{n(k)})_k$ conv. in X (pre) / U (compact)

(Bolzano-Weierstraß)

A pre $\Rightarrow$ bounded

B pre $\Leftarrow$ bounded

C pre $\Leftrightarrow$ bounded $(\dim(X) < \infty)$

D nothing

compact $\Leftrightarrow$ every open cover has finite subcover

(Heine-Borel)

Arzelà-Ascoli

Let $G \subset \mathbb{R}^n$ be compact.

Theorem (Arzelà-Ascoli [Kress LIE 3rd ed. Thm. 1.18])

$U \subset C(G)$ is precompact iff it is bounded and equicontinuous.

$\Leftarrow$

Equicontinuous means

$\forall f \in U$

$\forall x,y \;\; \forall \varepsilon > 0 \;\; \exists \delta > 0$ so that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Continuous means:

$\forall x,y$ $\qquad$ $\forall x,y$ "uniform"

$\forall \varepsilon > 0 \;\; \exists \delta > 0$ so that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Arzelà-Ascoli: Proof Sketch for (b) $\wedge$ (e) $\Rightarrow$ precompact

Let $\mathcal{U} \subset C(G)$ equicont. and bounded. Let $(\varphi_n) \subset \mathcal{U}$.

To show: $\forall \varepsilon > 0 \ \exists K \ \forall k, \ell \geq K \ \|\varphi_{n(k)} - \varphi_{n(\ell)}\|_\infty \leq \varepsilon$.

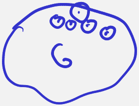
$n: \mathbb{N}_0 \to \mathbb{N}_0$

Let $\varepsilon > 0$. From equi cont., there exists a $\delta > 0$. Use that.

Consider cover $\{ B(x, \delta) : x \in G \}$

G compact: There exists an N so that

$$G \subseteq B(x_1, \delta) \cup B(x_2, \delta) \cup \dots \cup B(x_N, \delta).$$



$\{ \varphi_n(x) : n \in \mathbb{N}_0 \}$ is bounded $\Rightarrow$ precompact

$\Rightarrow$ can pick conv. subseq out of $(\varphi_n(x))_n$

Arzelà-Ascoli: Proof Sketch for b∧ e ⇒ precompact

By taking subseq. of subseq., get $(\varphi_{n(k)})_k$ that converges at all of them.

Let $x \in G$. There exists an i so that $\|x_i - x\| < \delta$.

$$
\begin{aligned}
&\|\varphi_{n(k)}(x) - \varphi_{n(\ell)}(x)\| \\
&\le \|\varphi_{n(k)}(x) - \varphi_{n(k)}(x_i)\| && < \varepsilon \quad \text{equicont.} \\
&+ \|\varphi_{n(k)}(x_i) - \varphi_{n(\ell)}(x_i)\| && (\varphi_{n(k)})_k \text{ is pw. Cauchy } \exists N \; < \varepsilon \\
&+ \|\varphi_{n(\ell)}(x_i) - \varphi_{n(\ell)}(x)\| && < \varepsilon \quad \text{equicont.} \\
&< 3\varepsilon
\end{aligned}
$$

Arzelà-Ascoli (II)

Intuition?

"prevents function values from running away"

"Uniformly continuous"?

When does *uniform continuity* happen?

continuous func on a compact

(Note: Kress LIE 2nd ed. defines 'uniform equicontinuity' in one go.)

Outline

Integral Operators are Compact

Theorem (Continuous kernel ⇒ compact [Kress LIE 3rd ed. Thm. 2.28])

$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then

$$(A\phi)(x) := \int_G K(x,y)\phi(y)dy.$$

is compact on $C(G)$.

Use A-A. (a statement about compact *sets*) What is there to show?
Pick $U \subset C(G)$. $A(U)$ bounded?

bounded

Let $\|\varphi\|_\infty \le C$. $\quad \|A\varphi(x)\| \le \|\int K(x,y)\,\varphi(y)\,dy\|$

$\le \int |K(x,y)|\,|\varphi(y)|\,dy$

$\le \int |K(x,y)|\,dy\,\|\varphi\|_\infty$

$\le \max|K| \cdot |G| \cdot \|\varphi\|_\infty$

$A(U)$ equicontinuous?

For Ec:

To show $\forall \varepsilon > 0\ \exists \delta > 0\ \forall \varphi$

Let $\varepsilon > 0$.

$$|x-y| < \delta \Rightarrow |A\varphi(x) - A\varphi(y)| < \varepsilon.$$

$$\cancel{|\varphi(x) - \varphi(y)| < \varepsilon}$$

$$A(U) = \{ A\varphi : \|\varphi\|_\infty \leq c \}$$

$$|A\varphi(x) - A\varphi(y)| = \left| \int (k(x,z) - k(y,z))\, \varphi(z)\, dz \right|$$

k is unif. cont. Ask it for the δ that works for all x.

$$\leq \int_G \varepsilon\, |\varphi(z)|\, dz \leq \varepsilon\, |G|\, \|\varphi\|_\infty .$$

$\Rightarrow A(U)$ equicont.

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition (Weakly singular kernel)

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n]$ such that

$$|K(x, y)| \leq C|x - y|^{\alpha - n} \qquad (x \neq y)$$

$\frac{1}{r^{1.999\ldots}}$

Ok in $\mathbb{R}^2$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 3rd ed. Thm. 2.29])

K weakly singular. Then

$$(A\phi)(x) := \int_G K(x, y)\phi(y)dy.$$

is compact on $C(G)$, where $\mathrm{cl}(G^\circ) = G$.

Weakly singular: Proof Outline

Outline the proof of 'Weakly singular kernel $\Rightarrow$ compact'.

Weakly singular (on surfaces)

$\Omega \subset \mathbb{R}^n$ bounded, open, $\partial\Omega$ is C^1 (what does that mean?)

Definition (Weakly singular kernel (on a surface))

- K defined, continuous everywhere except at $x = y$
- There exist $C > 0$, $\alpha \in (0, n-1]$ such that

$$|K(x, y)| \le C|x - y|^{\alpha - n + 1} \qquad (x, y \in \partial\Omega,\ x \ne y)$$

$\frac{1}{r^{0.999}}$

OK in $\mathbb{R}^2$

Theorem (Weakly singular kernel $\Rightarrow$ compact [Kress LIE 3rd ed. Thm. 2.30])

K weakly singular on $\partial\Omega$. Then $(A\phi)(x) := \int_{\partial\Omega} K(x, y)\phi(y)dy$ is compact on $C(\partial\Omega)$.

Q: Has this estimate gotten worse or better?

Outline