# CS 598 EVS: Tensor Computations

## Tensor Eigenvalues

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# Matrix Eigenvalues

- The eigenvalue and singular value decompositions of matrices enable not only low-rank approximation (which we can get for tensors via decomposition), but also describe important properties of the matrix $\boldsymbol{M}$ and associated linear function $\boldsymbol{f}^{(\boldsymbol{M})}(\boldsymbol{x}) = \boldsymbol{M}\boldsymbol{x}$

# Tensor Eigenvalues

- Tensor eigenvalues and singular values can be defined based on the function $\boldsymbol{f}^{(\mathcal{T})}$ by analogy from the role of matrix eigenvalues on $\boldsymbol{f}^{(\boldsymbol{M})}$

# Matrix Eigenvalues and Critical Points

- The eigenvalues/eigenvectors of a matrix are the critical values/points of its Rayleigh quotient[1]

- Singular vectors and singular values of matrices may be derived analogously

[1] Lek-Heng Lim, "Singular Values and Eigenvalues of Tensors: A Variational Approach", 2005

# Tensors Eigenvalues

- The Lagrangian approach to matrix eigenvalues generalizes naturally to symmetric tensors

# Tensor Singular Values and Singular Vectors

- Tensor singular values again can be viewed as critical points of the Lagrangian function of the multilinear map given by a tensor

# Immediate Properties of Tensor Eigenvectors and Singular Vectors

- When the tensor order $d$ is odd, $H$-eigenvectors ($l^d$-eigenvectors) and singular vectors must be defined with additional care
- The largest tensor singular value is the operator/spectral norm of the tensor

# Eigenvalues of Nonsymmetric Tensors

- For nonsymmetric matrices case, the Lagrangian approach used above cannot be used to describe the eigenvalues

# Another View of Symmetric Tensor Eigenvalues

- We can characterize eigenvectors with $\lambda = 0$ by considering the polynomial associated with a symmetric tensor, and similar with others.[2]

[2] Cartwright, Dustin, and Bernd Sturmfels. "The number of eigenvalues of a tensor." Linear algebra and its applications 438.2 (2013): 942-952.

# Number of eigenvalues in a tensor

- The number of eigenvalues over $\mathbb{C}$ can be derived from the view of polynomial equations

- In the real case, eigenvalues need not be real

# Example of Symmetric Tensor with Infinite Eigenvalues

- Concretely, the following symmetric tensor has infinite eigenvalues if they are normalized as $\boldsymbol{x}^H\boldsymbol{x}$[3]

$$a_{111} = 2, a_{122} = a_{212} = a_{221} = a_{133} = a_{313} = a_{331} = 1$$

and otherwise $a_{ijk} = 0$

[3] Cartwright, Dustin, and Bernd Sturmfels. "The number of eigenvalues of a tensor." Linear algebra and its applications 438.2 (2013): 942-952.

# Connection Between Decomposition and Eigenvalues

- In the matrix-case, the largest magnitude eigenvalue and singular value may be associated with a rank-1 term that gives the best rank-1 decomposition of a matrix

- In the tensor case, the rank-1 approximation problem corresponds to a maximization problem[4]

[4] L. De Lathauwer, B. De Moor, and J. Vandewalle, "On the best rank-1 and rank-($R_1$, $R_2$,..., $R_n$) approximation of higher-order tensors", 2000

# Derivation of Equivalence

- The singular value problem can be derived from decomposition via the method of Lagrange multipliers

# Hardness of Eigenvalue Computation

- Like rank-1 approximation, computing eigenvalues of singular values of a tensor is NP-hard, which can be demonstrated by considering the tensor bilinear feasibility problem[5]

[5] C.J. Hillar and L.-H. Lim, "Most tensor problems are NP-hard", 2013

# Hardness of Eigenvalue Computation

- NP-hardness of the tensor bilinear feasibility problem can be demonstrated by reduction from 3-colorability

# Power Method for Singular Value Computation

- The *high-order power method (HOPM)* can be used to compute the largest singular value[6]

[6] L. De Lathauwer, B. De Moor, and J. Vandewalle, "On the best rank-1 and rank-($R_1$, $R_2$,..., $R_n$) approximation of higher-order tensors", 2000

# Power Method for Symmetric Eigenvalue Problems

- The HOPM algorithm can be adapted to symmetric tensors

# Newton-based Methods for Eigenvalue Computation

- A state-of-the-art method of Newton-type method Newton Correction Method (NCM) for computing real eigenvectors of a symmetric tensor[7]

[7] Jaffe, Ariel, Roi Weiss, and Boaz Nadler. "Newton correction methods for computing real eigenpairs of symmetric tensors." SIAM Journal on Matrix Analysis and Applications 39.3 (2018).

# Perron-Frobenius Theorem for Tensor Eigenvalues

- The Perron-Frobenius theorem states that positive matrices have a unique real eigenvalue and the associated eigenvector is positive
- Tensor eigenvalues satisfy a generalized Perron-Frobenius theorem

# Tensor Eigenvalues and Hypergraphs

- Matrix eigenvalues are prominent in algebraic graph theory
- Tensor eigenvalues can be used to understand partitioning/clustering properties of uniform hypergraphs[8]

[8] J. Chang, Y. Chen, L. Qi, H. Yan, "Hypergraph Clustering Using a New Laplacian Tensor with Applications in Image Processing", 2019